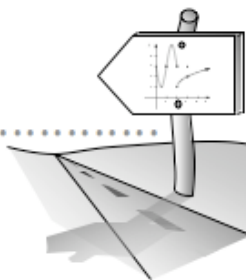# 1.2.1 What if the function is in pieces?

Piecewise Functions and Continuity

1-11. On the same axis, graph these two functions:

$g(x) = \frac{1}{2}x + 1$ for x < 4, and h(x) = −x + 6 for x ≥ 4.

   a. What happens to the graph at $x = 4$?

   b. When we combine parts of several functions to make a single function, we call it a **piecewise function.** Just as the graphs can be drawn on the same set of axes, the algebraic functions of g and h can be "pieced together" as a single function. If using a TI calculator or Desmos Graphing Calculator, refer to their links for appropriate videos inputting piecewise functions.

   We will call this function $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 4 \\ -x + 6, & x \geq 4 \end{cases}$

   Evaluate: $f(0)$, $f(4)$, and $f(6)$.
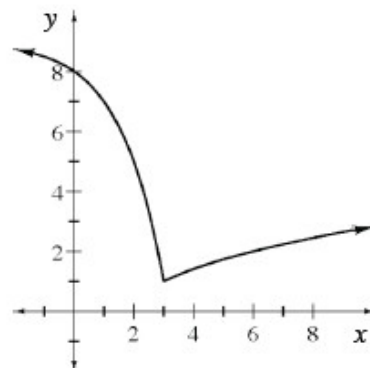

MATH NOTES

Intuitive Notion of Continuity

We will formally define continuity later in Chapter 2. For now, we can say that a function is **continuous** if you can draw the graph of the function without lifting your pencil from the paper.

The graph of a continuous function is shown at right.

The equation of this piecewise functions is $y = \begin{cases} 9 - 2^x & \text{for } x \leq 3 \\ \sqrt{x-2} & \text{for } x > 3 \end{cases}$.

The graph is continuous at $x = 3$ because $9 - 2^3 = \sqrt{3-2}$.

Though the graph is connected at $x = 3$, notice that the domains of the individual equations do NOT overlap at $x = 3$. This ensures that the piecewise is a function.

When the y-values of a function are not connected as in problem 1-11, we say that the function is **discontinuous.** The point of discontinuity is the x-value where the function becomes discontinuous. The point of discontinuity in problem 1-11 is $x = 4$.

1-12. Examine the graph of the function $h(x) = \begin{cases} 9 - 2^x & \text{for } x \le 3 \\ \sqrt{x-2} & \text{for } 3 < x \end{cases}$ in the Math Note above.

   a. Is $h(x)$ continuous or discontinuous? Justify your answer.

   b. Explain why $h(x)$ could also be defined as $h(x) = \begin{cases} 9 - 2^x & \text{for } x < 3 \\ \sqrt{x-2} & \text{for } 3 \le x \end{cases}$.

MATH NOTES

Domain and Range Notation

There are two accepted forms of notation for domain and range. Examine the graph below. The domain and range of the function shown can be noted in either **interval** notation or in **set** notation.

Interval Notation:

$D = [0,1] \cup (1, \infty)$

$R = [1, \infty)$

Set Notation:
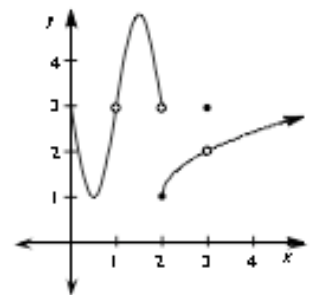
$D = \{x : x \ge 0 \text{ and } x \ne 1\}$
$R = \{y : y \ge 1\}$

OR

$D = \{x : 0 \le x < 1 \text{ or } x > 1\}$
$R = \{y : y \ge 1\}$

1-13. Explain why the two forms of set notation for the domain, D, in the Math Notes box above are equivalent.
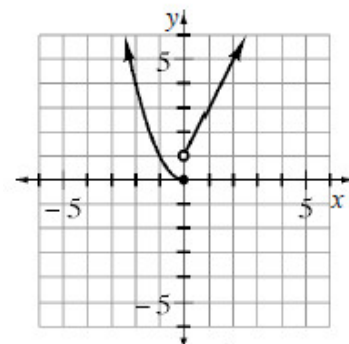
1-14. Compare the domains of $f(x)$, $g(x)$, and $h(x)$ below. Explain what aspect of each function limits the domain.

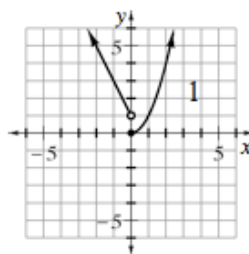$f(x) = \sqrt{x - 25}$
$g(x) = \frac{1}{x-25}$
$h(x) = \log(x - 25)$

1-15. Given $f(x) = \begin{cases} x^2 & \text{for } x \le 0 \\ 2x+1 & \text{for } x > 0 \end{cases}$ and its graph at right, match each transformation to its graph. Test your ideas using the 1-15 Student eTool.
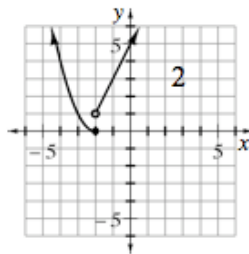
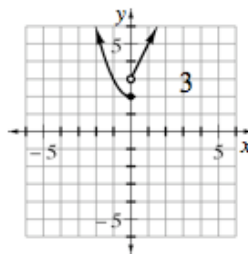|	
A. $f(x+2) = \begin{cases} (x+2)^2 & \text{for } x \le -2 \\ 2(x+2)+1 & \text{for } x > -2 \end{cases}$	1.
B. $f(x)+2 = \begin{cases} x^2+2 & \text{for } x \le 0 \\ 2x+3 & \text{for } x > 0 \end{cases}$	2.
C. $f(-x) = \begin{cases} (-x)^2 & \text{for } x \ge 0 \\ 2(-x)+1 & \text{for } x < 0 \end{cases}$	3.

a. Find the domain and range for $f(x)$.

b. At what value of x is $f(x)$ is discontinuous?

c. At what value of x is $f(x+2)$ discontinuous? What about $f(x)+2$? Explain the different answers.

1-16. There is a debate among mathematicians over whether $f(x) = |x|$ should be considered a parent graph, or if it is simply a piecewise function such that: $f(x) = |x| = \begin{cases} -x & \text{for } x \le 0 \\ x & \text{for } x > 0 \end{cases}$.

a. Write $g(x) = |x-2|$ as a piecewise function.

b. Write $h(x) = \begin{cases} -(x+5) & \text{for } x \le -5 \\ x+5 & \text{for } x > -5 \end{cases}$ as an absolute value function.

c. Why do you think this issue is *debatable*?

1-17. Given the function $f(x) = \begin{cases} x^2 + 2 & \text{for } x < 1 \\ -x & \text{for } x \geq 1 \end{cases}$

a. Sketch the graph of $f(x)$.

b. Modify one piece of $f(x)$ so that the function is continuous.

1-18. Find values of a and b so that $g(x)$ is continuous. How does the domain help you to solve this algebraically? Use the 1-18 Student eTool to verify your ideas.

$$g(x) = \begin{cases} \sqrt{x+3} & \text{for } x < 1 \\ a(x-1)^2 + b & \text{for } 1 \leq x < 3 \\ -x + 2 & \text{for } x \geq 3 \end{cases}$$

1-19. Selected values of a continuous function, $f(x)$, are shown in the table below.

x	−4	−2	0	2	4	6
f(x)	7	5	3	1	1	3

a. If the graph of $f(x)$ has one unique minimum value, what do you think this value is? Explain your thinking.

b. Could the graph of $f(x)$ be a parabola? If so, find a possible equation of $f(x)$. If not, explain.

c. Could the graph of $f(x)$ be an absolute value function? If so, find a possible equation of $f(x)$. If not, explain.

d. Is it possible that $f(3) = 10$. Explain.

e. Is it possible that $f(x)$ has a vertical asymptote at $x = 3$? Explain.

Review & Preview

1-20. Let $g(x) = \begin{cases} 2 & \text{for } 0 \leq x \leq 3 \\ 3 & \text{for } 3 < x \leq 5 \\ 7 & \text{for } 5 < x \leq 8 \end{cases}$ Help (Html5)⇔Help (Java)

a. Sketch the graph of $g(x)$. Is this function continuous? Explain.

b. Shade the area between $g(x)$ and the x-axis. Find $A(g, 0 \leq x \leq 8)$.

c. The equation $g(x)$ is an example of a **step function.** Why do you think it is called a step function?

1-21. Given the following functions, compute the given values.   Help (Html5)⇔Help (Java)

a. $f(x) = \begin{cases} 4-3x & \text{for } x \le 1 \\ x^2 & \text{for } x > 1 \end{cases}$ Find $f(0)$, $f(1)$, and $f(3)$.

b. $f(x) = \begin{cases} \sqrt{x} & \text{for } x < 3 \\ 3-x & \text{for } x \ge 3 \end{cases}$ Find $f(1)$, $f(3)$, and $f(9.4)$.

c. $f(x) = \begin{cases} -x & \text{for } x \le 0 \\ \frac{5}{x} & \text{for } 0 < x \le 1 \\ 6-2x & \text{for } x > 1 \end{cases}$ Find $f(-3)$, $f(0)$, $f(0.5)$ and $f(4)$.

d. Sketch a graph of $f(x)$ in part (c) above.

1-22. In order to mail a letter in the United States, postage must be paid based on the weight of the letter. Although rates are tied to the number of ounces, the U. S. Post Office does not allow for payments of partial ounces. For example, if your letter weighs 6.1 ounces, you must pay for 7 ounces.

A graph showing the postage rates for letters weighing fewer than five ounces for the year 2009 is shown below. This is another example of a step function.   Help (Html5)⇔Help (Java)
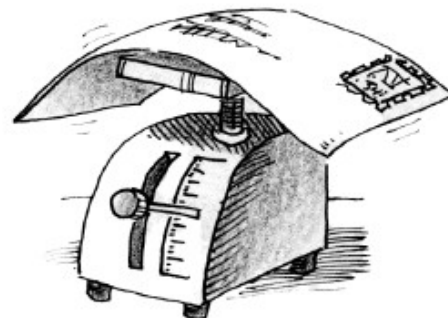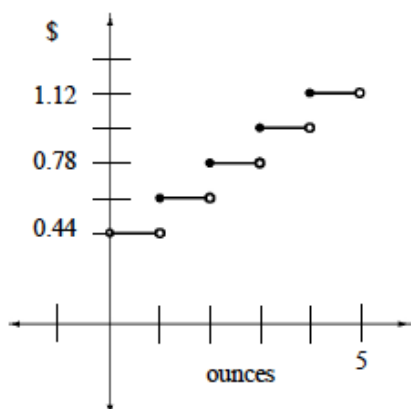
a. How much would you pay for a letter weighing 2.9 ounces? For 3 ounces? For 3.1 ounces?

b. Write a piecewise function that determines the postage rates for letters weighing between 0 and 5 ounces. Let x represent ounces, and y represent cost.

**1-23.** A semi-circular flag is shown attached to a "pole" below.   Help (Html5)⇔Help (Java)
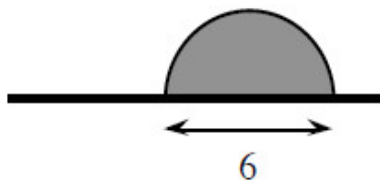
a. Imagine rotating the flag about its pole and describe the resulting three-dimensional figure. Draw a picture of this figure on your paper.

b. Find the volume of the rotated flag.

1-24. Sketch the graph of the piecewise function $g(x)$ below.   Help (Html5)⇔Help (Java)

$$g(x) = \begin{cases} 2^x & \text{for } x \leq 2 \\ 3x - 2 & \text{for } x > 2 \end{cases}$$

    a. State the domain and range of $g(x)$.

    b. Is $g(x)$ continuous at $x = 2$? Explain.

    c. Is $g(x)$ continuous on all values of x?

1-25. The parabola $y = -(x - 3)^2 + 4$ is graphed below. Four trapezoids of equal width are inscribed for $1 \leq x \leq 5$.   Help (Html5)⇔Help (Java)
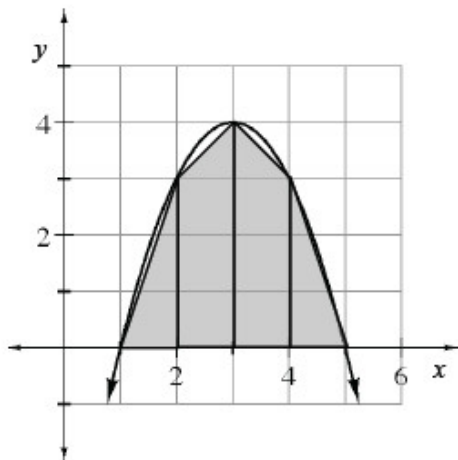
    a. Use the combined area of these trapezoids to approximate the area under the parabola for $1 \leq x \leq 5$.

    b. Is this area higher or lower than the true area under the parabola? Explain.

1-26. Find the exact value of each of the following trig expressions.   Help (Html5)⇔Help (Java)

    a. $\sin \frac{5\pi}{3}$

    b. $\tan \frac{7\pi}{6}$

    c. $\sec \frac{5\pi}{4}$

    d. $\csc \pi$

1-27. Sketch the graph of $y = \frac{1}{x}$.   Help (Html5)⇔Help (Java)

    a. Why does this graph have a vertical asymptote? What is the equation of that asymptote?

    b. State the equation of the horizontal asymptote.

    c. Alter $y = \frac{1}{x}$ so that the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = 3$.

MATH NOTES

Polynomial Division with Remainders

The examples below show two methods for dividing polynomials. Both have remainders that we write as fractions.

$$\frac{x^4 - 6x^3 + 18x - 1}{x - 2}$$

Using Long division:

$$
\begin{array}{r}
x^3 - 4x^2 - 8x + 2 \\
x - 2 \overline{)\, x^4 - 6x^3 + 0x^2 + 18x - 1} \\
\underline{x^4 - 2x^3} \\
-4x^3 + 0x^2 \\
\underline{-4x^3 + 8x^2} \\
-8x^2 + 18x \\
\underline{-8x^2 + 16x} \\
2x - 1 \\
\underline{2x - 4} \\
3
\end{array}
$$

Remainder

Final Answer: $x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}$

Using Generic Rectangles:

Remainder

	x^3	$-4x^2$	$-8x$	$+2$	
x	x^4	$-4x^3$	$-8x^2$	$+2x$	3
-2	$-2x^3$	$+8x^2$	$+\ 16x$	-4	

$$x^4 \quad -6x^3 \quad +0x^2 \quad +18x \quad -1$$

Final Answer: $x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}$

1-28. Use polynomial division to simplify the following rational expressions.

a. $\dfrac{x^3+2x^2-3x+4}{x+3}$

b. $\dfrac{x^4-5x^2+3x-3}{x-2}$