Section 2.3 Polynomial Functions and Their Graphs

Question: Just do a summary: What did we do in this lesson? What do you still need help with? ...

Warm-Up

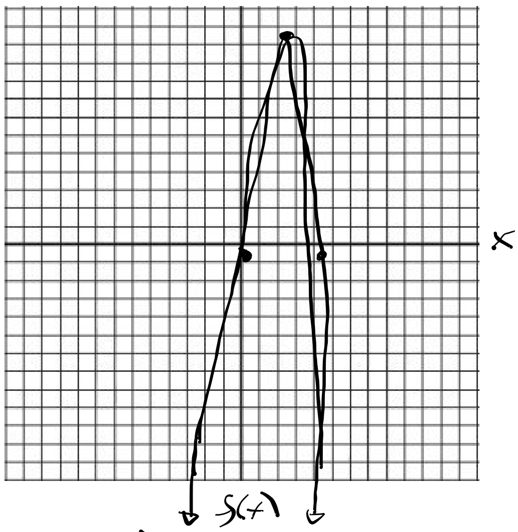
$a(x-h)^2+k$

Graph $f(x)=-3(x-2)^2+12$

Which way does it open? down

Vertex: $(2,12)$

(h,K)



x-int $f(x)=0$

$0=-3(x-2)^2+12$

$-12=-3(x-2)^2$

$4=(x-2)^2$

$\pm 2=x-2$

$2=x-2$ $\qquad$ $-2=x-2$

$4=x$ $\qquad$ $0=x$

y-int $f(0)$

$f(0)=-3(0-2)^2+12$

$=-3(-2)^2+12$

$=-12+12=0$

Smooth, Continuous Graphs

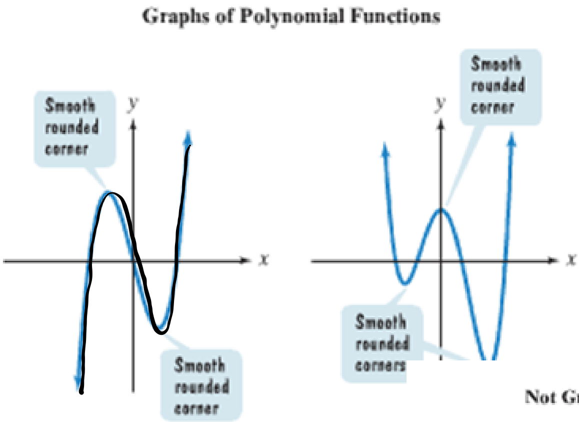
Definition of a Polynomial Function

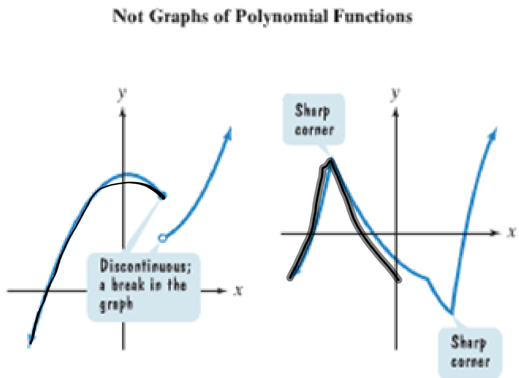
Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers, with $a_n \neq 0$. The function defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of degree n**. The number a_n, the coefficient of the variable to the highest power, is called the **leading coefficient**.

Polynomial functions of degree 2 or higher have graphs that are smooth and continuous. By **smooth**, we mean that the graphs contain only rounded curves with no sharp corners. By **continuous**, we mean that the graphs have no breaks and can be drawn without lifting your pencil from the rectangular coordinate system.





Notice the breaks and lack of smooth curves.

End Behavior of Polynomial Functions

The Leading Coefficient Test

As x increases or decreases without bound, the graph of the polynomial function

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 \quad (a_n \neq 0)$$

eventually rises or falls. In particular,

1. For n odd:

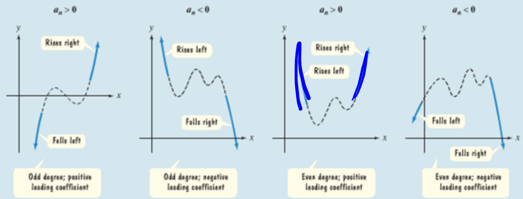
If the leading coefficient is positive, the graph falls to the left and rises to the right.

If the leading coefficient is negative, the graph rises to the left and falls to the right.

2. For n even:

If the leading coefficient is positive, the graph rises to the left and to the right.

If the leading coefficient is negative, the graph falls to the left and to the right.

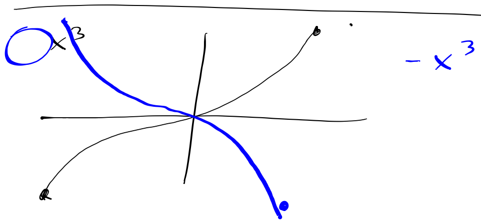


degree

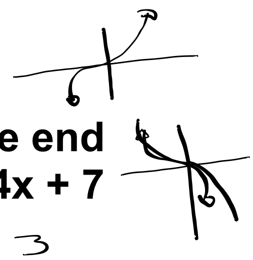
$2x^5 + x^4 + x^3 + 2x^2 - 4$

degree 5

leading coefficient



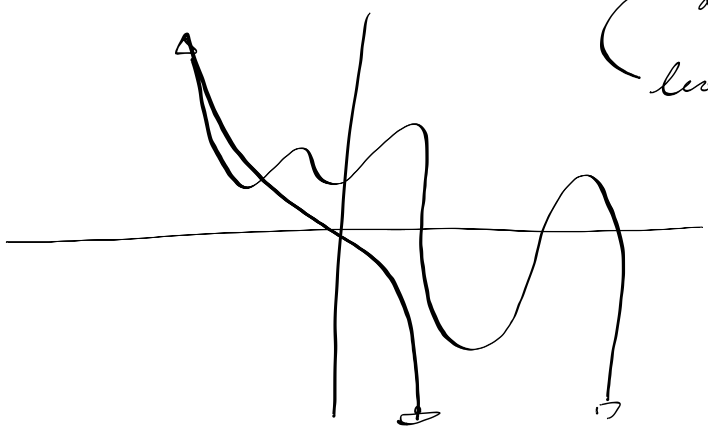
Odd-degree polynomial functions have graphs with opposite behavior at each end. Even-degree polynomial functions have graphs with the same behavior at each end.



Example

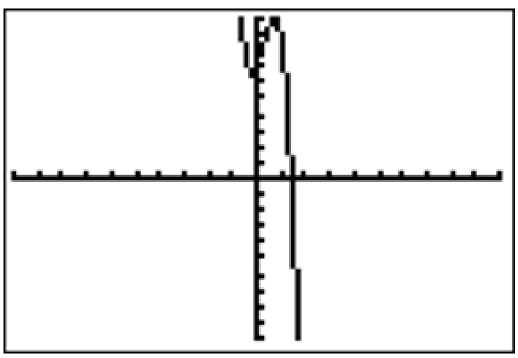
Use the Leading Coefficient Test to determine the end behavior of the graph of $f(x) = -3x^3 - 4x + 7$

degree = 3
ODD
leading coefficient = −3



Example

Use the Leading Coefficient Test to determine the end behavior of the graph of $f(x) = -.08x^4 - 9x^3 + 7x^2 + 4x + 7$
This is the graph that you get with the standard viewing window. How do you know that you need to change the window to see the end behavior of the function? What viewing window will allow you to see the end behavior?



Zeros of Polynomial Functions

roots

x-ints

where $f(x)=0$

If f is a polynomial function, then the values of x for which f(x) is equal to 0 are called the **zeros** of f. These values of x are the **roots**, or **solutions**, of the polynomial equation f(x)=0. Each real root of the polynomial equation appears as an x-intercept of

$f(x)=0$

Find all zeros of $f(x)= x^3+4x^2- 3x - 12$

By definition, the zeros are the values of x for which f(x) is equal to 0. Thus we set f(x) equal to 0 and solve for x as follows:

~~$(x^3+4x^2)+(3x+12)=0$~~ $(x^3+4x^2)(-3x-12)$

$x^2(x+4)-3(x+4)=0$

$a \cdot b=0$

$a=0$ or $b=0$

$(x+4)(x^2-3)=0$

$x+4=0 \quad x^2-3=0$

$x=-4 \qquad x^2=3$

$x=\pm\sqrt{3}$

Example

Find all zeros of $x^3+2x^2-4x-8=0$. But also show me the end behavior, please

$f(x)=0$

$0=(x^3+2x^2)(-4x-8)$

$0=x^2(x+2)-4(x+2)$

$0=(x^2-4)(x+2)$

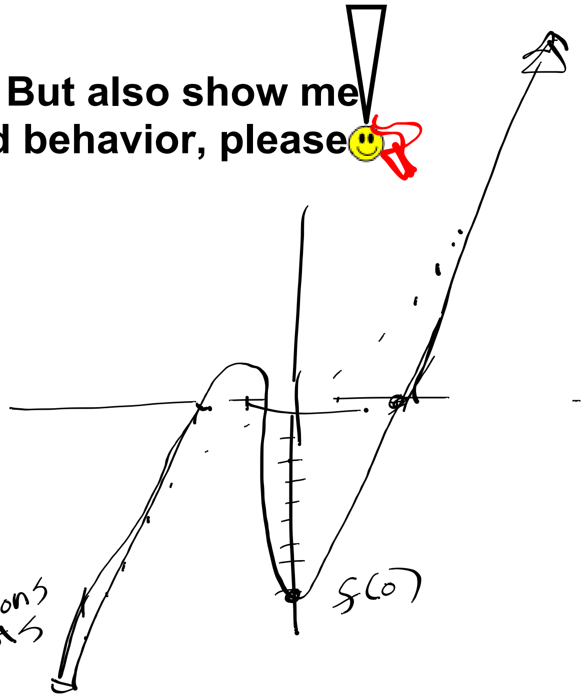
$x^2-4=0 \qquad x+2=0$

$x^2=4 \qquad x=-2$

$x=\pm 2$

$x=2 \quad x=-2 \quad x=-2$

zero
solutions
roots



$x=-2$

multiplicity of 2

Multiplicity of x-Intercepts

Multiplicity and *x*-Intercepts

If *r* is a zero of **even multiplicity**, then the graph **touches** the *x*-axis **and turns around** at *r*. If *r* is a zero of **odd multiplicity**, then the graph **crosses** the *x*-axis at *r*. Regardless of whether the multiplicity of a zero is even or odd, graphs tend to flatten out at zeros with multiplicity greater than one.

For f(x)= $x^2(x-2)^2$, notice that each factor occurs twice. In factoring this equation for the polynomial function f, if the same factor x-r occurs k times, but not k+1 times, we call r a zero with multiplicity k. For the polynomial above both 0 and 2 are zeros with multiplicity 2.

Find the zeros of $x^3+2x^2-4x-8=0$

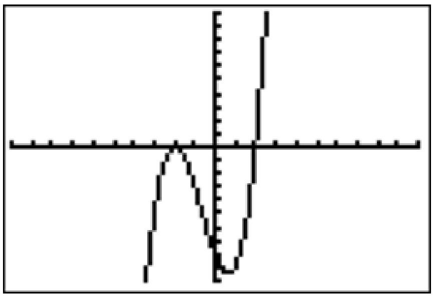
$$\left(x^3+2x^2\right)+\left(-4x-8\right)=0$$

$$x^2\left(x+2\right)-4(x+2)=0$$

$$\left(x+2\right)\left(x^2-4\right)=0$$

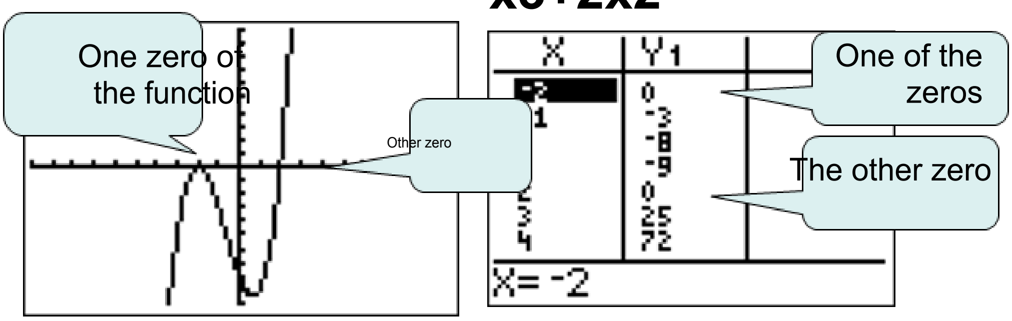
$$\left(x+2\right)\left(x+2\right)\left(x-2\right)=0$$

-2 has a multiplicity of 2, and 2 has a multiplicity of 1.
Notice how the graph touches at -2 (even multiplicity), but crosses at 2 (odd multiplicity).

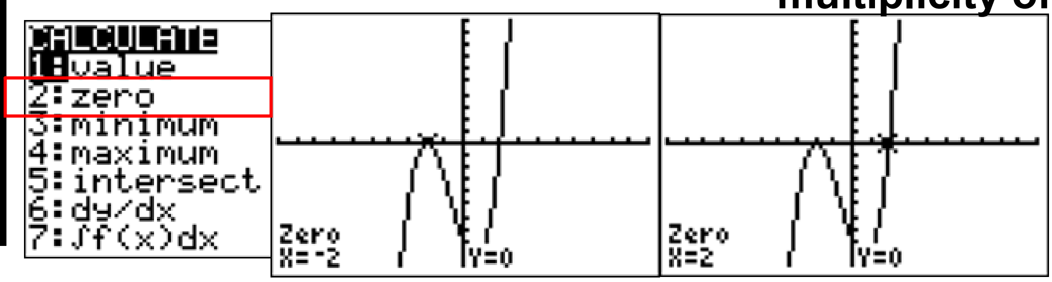


Graphing Calculator- Finding the Zeros

x3+2x2-



The x-intercepts are the zeros of the function. To find Trace then #2. The zero -2 has the zeros, press 2nd multiplicity of 2.



Example

$g(x)=(x-3)^2(x-1)^3(x+4)^1(x-16)^{20}$

Find the zeros of $f(x)=(x-3)^2(x-1)^3$ and give the multiplicity of each zero. State whether the graph crosses the x-axis or touches the x-axis and turns around at each zero.

$= (x-3)(x-3)(x-1)(x-1)(x-1)$

$f(x)=(x-3)^2(x-1)^3$

$0=(x-3)^2$ $\quad$ $0=(x-1)^3$

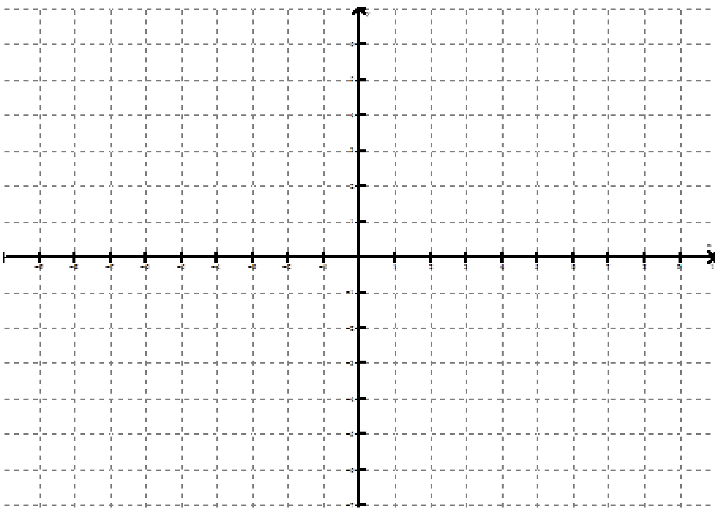
$x=3$ $\quad$ $x=1$

mult. $=2$ $\quad$ mult $=3$

Continued on the next

Example

Now graph this function on your calculator. $f(x)=(x-3)^2(x-1)^3$



The Intermediate Value Theorem

The Intermediate Value Theorem for Polynomials

Let f be a polynomial function with real coefficients. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of c between a and b for which $f(c) = 0$. Equivalently, the equation $f(x) = 0$ has at least one real root between a and b.

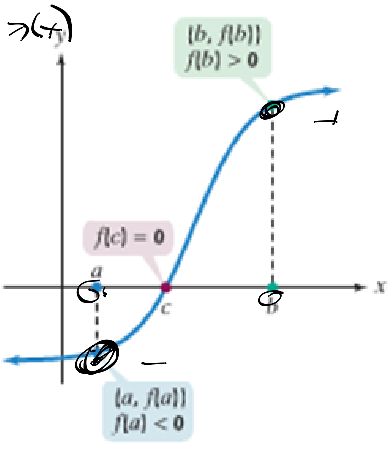
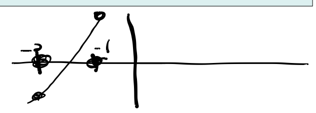


Figure 3.19 The graph must cross the x-axis at some value between a and b.

$2x^3 + 2x^2 - 3x + 4$

Show that the function $y=x^3-x+5$ has a zero between - 2 and -1.

$f(\overset{a}{-2})=(-2)^3-(-2)+5=-1$ $\quad -$



$f(\overset{b}{-1})=(-1)^3-(-1)+5=5$ $\quad +$

Since the signs of f(-1) and f(-2) are opposites then by the Intermediate Value Theorem there is at least one zero between f(-2) and f(-1). You can also see these values on the table below. Press 2nd Graph to get the table below.

X	Y1	
-4	-55	
-3	-19	
-2	-1	
-1	5	
0	5	
1	5	
2	11	

X= -4

Example

Show the polynomial function $f(x)=x^3-2x+9$ has a real zero between - 3 and - 2.

$f(-3)=(-3)^3-2(-3)+9$
$= -27+6+9$
$= -12$ → negative

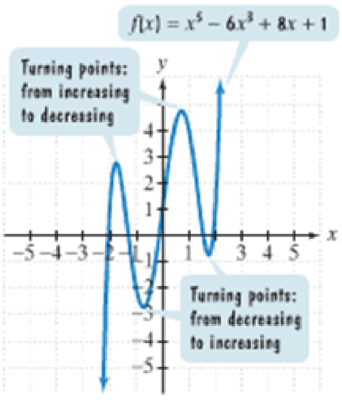
$f(-2)=(-2)^3-2(-2)+9$
$= -8+4+9$
$= 5$ → positive

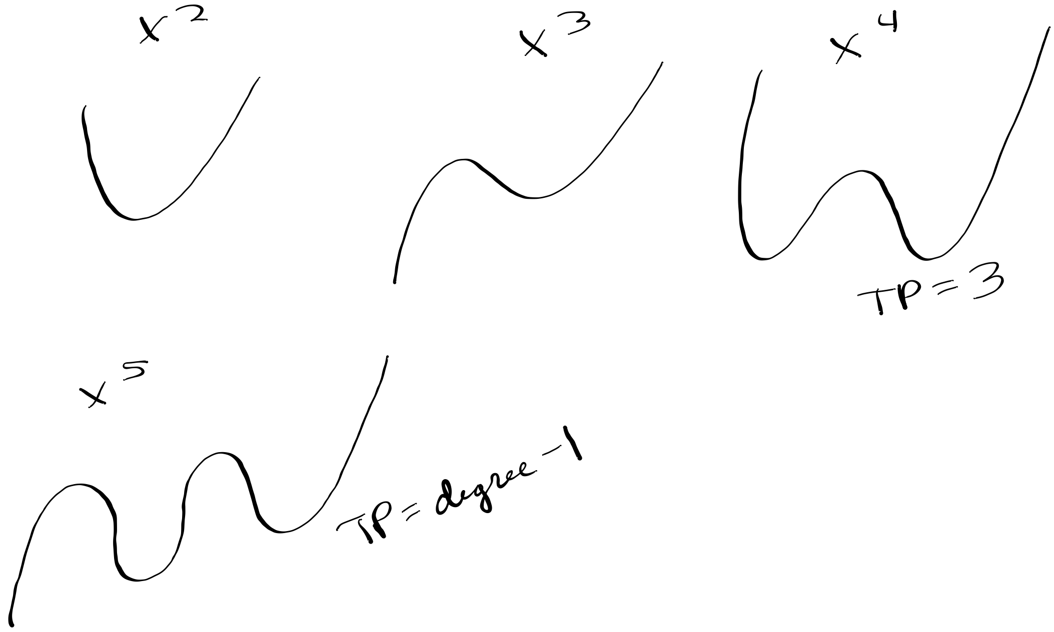
By the IVT $f(-3)= -$ and $f(-2) = +$
∴ there is a root btw -3 and -2

Turning Points of Polynomial functions

The graph of $f(x)=x^5-6x^3+8x+1$ is shown below. The graph has four smooth turning points. The polynomial is of degree 5. Notice that the graph has four turning points. In general, if the function is a polynomial function of degree n, then the graph has at most n-1 turning points.

x^7





A Strategy for Graphing Polynomial Functions

Graphing a Polynomial Function

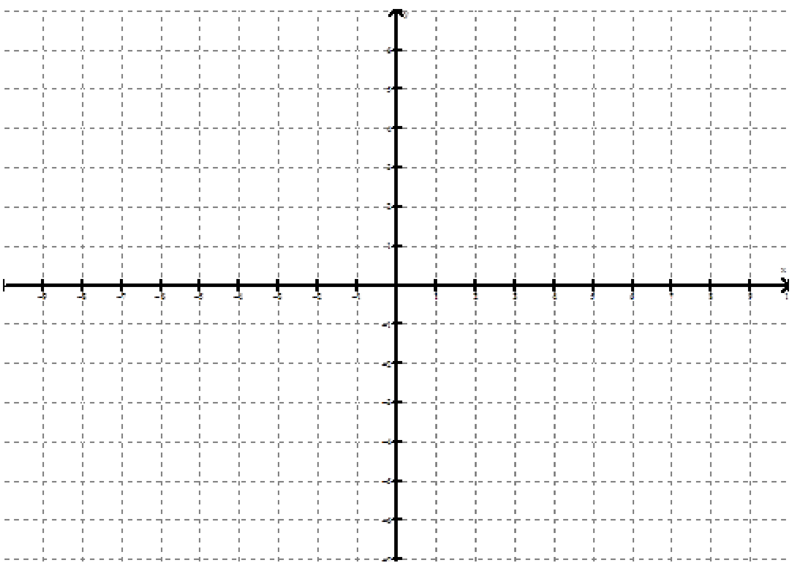
$3x^7 + 4x^6 - 5x^5 + 32x^4 \ldots$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,\ a_n \neq 0$$

1. Use the Leading Coefficient Test to determine the graph's end behavior.
2. Find x-intercepts by setting $f(x) = 0$ and solving the resulting polynomial equation. If there is an x-intercept at r as a result of $(x - r)^k$ in the complete factorization of $f(x)$, then
   - **a.** If k is even, the graph touches the x-axis at r and turns around.
   - **b.** If k is odd, the graph crosses the x-axis at r.
   - **c.** If $k > 1$, the graph flattens out at $(r, 0)$.
3. Find the y-intercept by computing $f(0)$.
4. Use symmetry, if applicable, to help draw the graph:
   - **a.** y-axis symmetry: $f(-x) = f(x)$
   - **b.** Origin symmetry: $f(-x) = -f(x)$.
5. Use the fact that the maximum number of turning points of the graph is $n - 1$ to check whether it is drawn correctly.

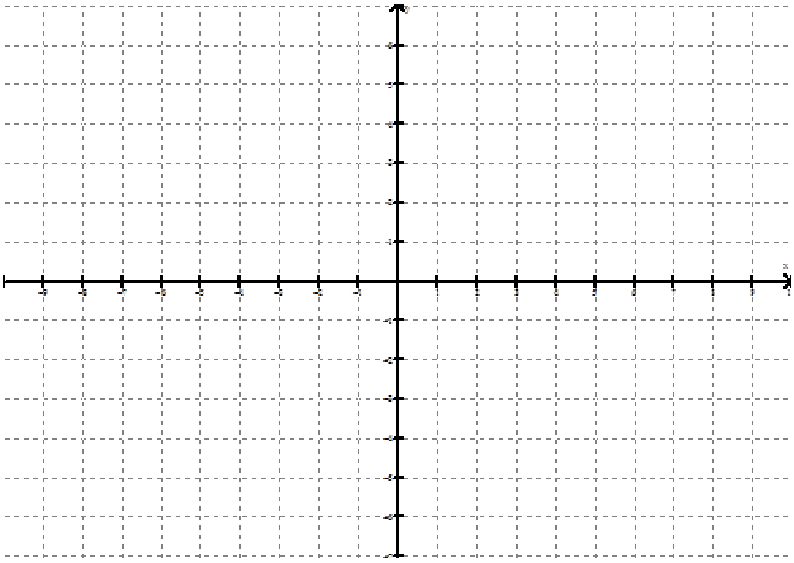
Example

Graph $f(x)=x^4 - 4x^2$ using what you have learned in this section.



Example

Graph $f(x)=x^3-9x^2$ using what you have learned in this section.

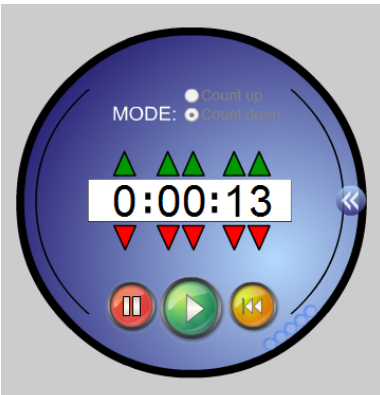


Use the Leading Coefficient Test to determine the end behavior of the graph of the polynomial function $f(x)=x^3-9x^2+27$

(a) falls left, rises right
(b) rises left, falls right
(c) rises left, rises right
(d) falls left, falls right

State whether the graph crosses the x-axis, or touches the x-axis and turns around at the zeros of 1, and - 3.
$f(x)=(x-1)^2(x+3)^3$

(a) -3 touches, 1 touches
(b) -3 crosses, 1 crosses
(c) -3 touches, 1 crosses
(d) -3 crosses, 1 touches



Due Thursday
t 2.3 - 2.4

p297-298 #15-63 odd
Wed

Part II. pp 310 #1-41 odd
1-15 Wed