Section 2.5
Zeros of Polynomial Functions

Question: What do the following theorems mean:

- Remainder Theorem
- Factor Theorem
- Intermediate Value Theorem
- Descartes Rule of Signs
- Rational Zeros Theorem
- Linear Factorization Theorem
- The Fundamental Theorem of Algebra

Extra Credit: Make Flashcards for these...

Solve $2x^3-5x^2+x+2=0$ given that 2 is a root.

By the factor theorem $x-2$ is a factor of $2x^3-5x^2+x+2=0$.

$(x-2)(\quad)$

$$\begin{array}{r|rrrr} 2 & 2 & -5 & 1 & 2 \\ & & 4 & -2 & -2 \\ \hline & 2 & -1 & -1 & 0 \end{array}$$

$(2x^2-x-1)(x-2)=0$

$(2x$

$2x^2-x-1=0$

$(2x^2-2x)+(1x-1)=0$

$2x(x-1)+1(x-1)=0$

$(2x+1)(x-1)=0$

$2x+1=0$ $\qquad$ $x-1=0$

$x=-\frac{1}{2}$ $\qquad$ $x=1$

-2, -2, 1, -1

$x-2=0$

$x=2$

$\boxed{2, -\frac{1}{2}, 1}$

The Rational Zero Theorem

Solve: $2x^3-5x^2+x+2=0$

The Rational Zero Theorem

If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has *integer* coefficients and $\frac{p}{q}$ (where $\frac{p}{q}$ is reduced to lowest terms) is a rational zero of f, then p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}}.$$

$2x^3-5x^2+x+2=0$

$p=2$ $1, 2, -1, -2$

$q=2$ $1, 2, -1, -2$

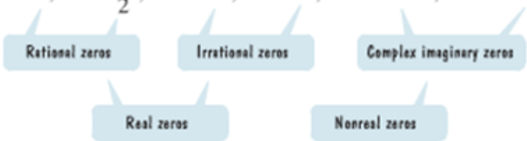
$\frac{-2}{-1}, \frac{-1}{2}, \frac{1}{2}, \frac{2}{-1}$

$\pm 2, \pm\frac{1}{2}$

Be sure you are familiar with the various kinds of zeros of polynomial functions. Here's a quick example:

$$f(x) = (x+3)(2x-1)(x+\sqrt{2})(x-\sqrt{2})(x-4+5i)(x-4-5i).$$

Zeros: -3, $\frac{1}{2}$, $-\sqrt{2}$, $\sqrt{2}$, $4-5i$, $4+5i$



Example

$q=1$ $p=12$

List all possible rational zeros of $f(x)=x^3-3x^2-4x+12$
Find one of the zeros of the function using synthetic division, then factor the remaining polynomial. What are all of the zeros of the function? How can the graph below help you find the zeros?

$\frac{p}{q} = \frac{\pm1, \pm2, \pm3, \pm4, \pm6, \pm12}{\pm1} = 1, 2, 3, 4, 6, 12$
$-1, -2, -3, -4, -6, -12$

$f(x) = x^3-3x^2-4x+12$

1) 1 -3 -4 12
1 -2 -6
1 -2 -6 6

$(x-2)(x^2-x-6)$

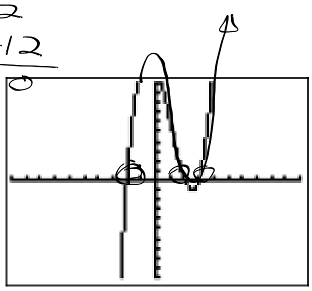
$(x-2)(x-3)(x+2)=0$

-1) 1 -3 -4 12
-1 4 0
1 -4 0 12

$\boxed{2, 3, -2}$

2) 1 -3 -4 12
2 -2 -12
1 -1 -6 0

$x^3 =$ 2 turns



Due Monday. We will not turn it in

Assignment: 2.5 part I

page 322 1-8 all

9-23 odd

yuh!

$\boxed{10-24 \text{ even}}$

QUIZZY on the rational zero theorem...

Solve: $2x^3-5x^2+x+2$

I want to see work...

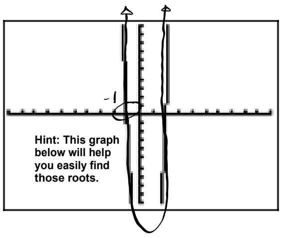
All the possible roots
Checking the possible roots
Box your answers

Example

List all possible rational zeros of $f(x)=6x^3-19x^2+2x+3$ Starting with the integers, find one zero of the function using synthetic division, then factor the remaining polynomial. What are all of the zeros of the function?

Warm-Up

List all possible rational roots of ***$x^4 - x^3 + 7x^2 - 9x - 18 = 0$*** **Starting with the integers, find two roots of the equation using synthetic division. Factor the remaining polynomial.** ***What are all of the roots of the equation?*** **The graph below will NOT help you find the imaginary roots. Why?**



possible roots $\pm 18, \pm 9, \pm 6, \pm 3, \pm 2, \pm 1$

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 7 & -9 & -18 \\ & & -1 & 2 & -9 & 18 \\ \hline & 1 & -2 & 9 & -18 & 0 \end{array}$$

$x^3 - 2x^2 + 9x - 18$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 9 & -18 \\ & & 2 & 0 & 18 \\ \hline & 1 & 0 & 9 & 0 \end{array}$$

$x^2 + 9$

$(x - (-1))(x - 2)(x^2 + 9) = 0$

$(x + 1)(x - 2)(x^2 + 9) = 0$

$x + 1 = 0$, $x = -1$

$x - 2 = 0$, $x = 2$

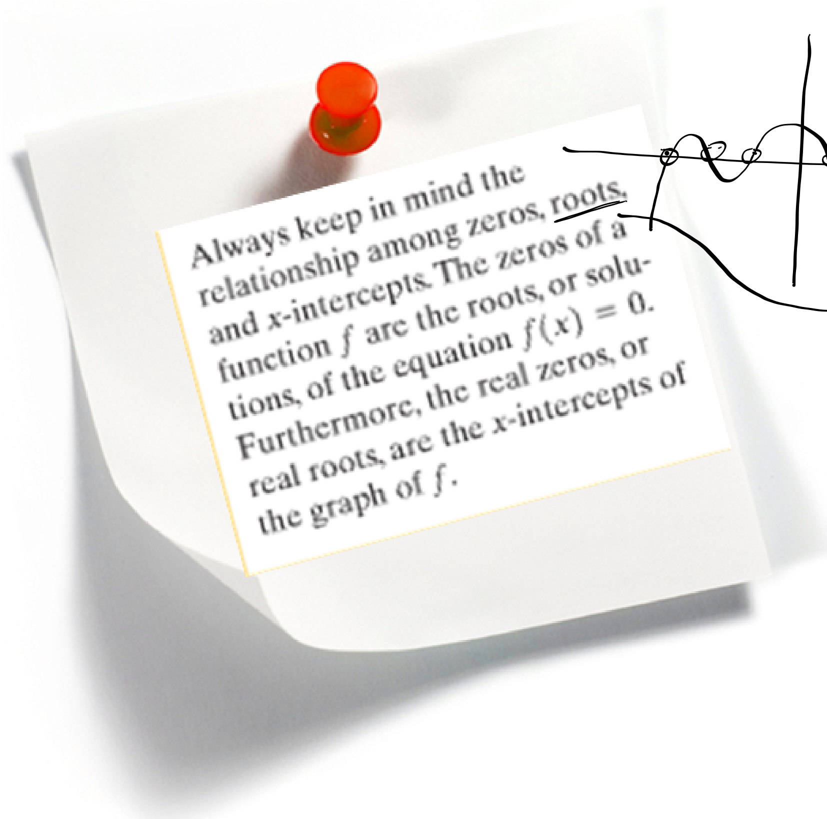
$x^2 + 9 = 0$

$x^2 = -9$

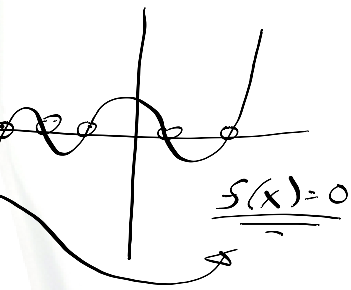
$x = \sqrt{-9}$

$x = \sqrt{(3i)^2}$

$x = \pm 3i$



Always keep in mind the relationship among zeros, roots, and x-intercepts. The zeros of a function f are the roots, or solutions, of the equation $f(x) = 0$. Furthermore, the real zeros, or real roots, are the x-intercepts of the graph of f.



Properties of Polynomial Equations

1. If a polynomial equation is of degree n, then counting multiple roots separately, the equation has n roots.
2. If $a + bi$ is a root of a polynomial equation with real coefficients ($b \neq 0$), then the complex imaginary number $a - bi$ is also a root. Complex imaginary roots, if they exist, occur in conjugate pairs.

conjugates: $3i$ $-3i$

$2-i$ $2+i$

$4+5i$ $4-5i$

Notice that the roots for our most recent problem

($x^4 - x^3 + 7x - 9x - 18 = 0$; degree 4) were $\pm 3i, 2, -1$

The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra
If $f(x)$ is a polynomial of degree n, where $n \geq 1$, then the equation $f(x) = 0$ has at least one complex root.

Remember that having roots of 3, -2, etc. are complex roots because 3 can be written 3+0i and -2 can be written as -2+0i.

The Linear Factorization Theorem

The Linear Factorization Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $n \geq 1$ and $a_n \neq 0$, then

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n),$$

where $c_1, c_2, \ldots, c_n$ are complex numbers (possibly real and not necessarily distinct). In words: An nth-degree polynomial can be expressed as the product of a nonzero constant and n linear factors.

$$f(x) = K(x - c_1)(x - c_2)(x - c_3)(x - c_4)$$

Example Find a fourth-degree polynomial function f(x) with real coefficients that has -1,2 and *i* as zeros and such that f(1)=- 4

$n = 4$

also $-i$

$$f(x) = K(x - c_1)(x - c_2)(x - c_3)(x - c_4)$$

$$f(x) = K(x+1)(x-2)(x-i)(x+i)$$

$$f(x) = K(x^2 - 2x + x - 2)(x^2 - ix + ix - i^2)$$

$i^2 = -1$

$-i^2 = 1$

$$f(x) = K(x^2 - x - 2)(x^2 + 1)$$

$$f(x) = K(x^4 + x^2 - x^3 - x - 2x^2 - 2)$$

$$f(x) = K(x^4 - x^3 - x^2 - x - 2)$$

$f(1) = -4$

$$-4 = K(1^4 - 1^3 - 1^2 - 1 - 2)$$

$$-4 = K(1 - 1 - 1 - 1 - 2)$$

$$-4 = K(-4)$$

$$1 = K$$

$$\boxed{f(x) = x^4 - x^3 - x^2 - x - 2}$$

Find a fourth-degree polynomial function f(x) with real coefficients that has -2,2 and *i* as zeros and such that f(3)=-150

Find a third-degree polynomial function f(x) with real coefficients that has -3 and *i* as zeros and such that f~~(1)=~~8 $f(1)=24$

$\sqrt{-1}\cdot\sqrt{-1}=-1$

$$f(x)=k(x-c_1)(x-c_2)(x-c_3)$$

$$f(x)=k(x+3)(x+i)(x-i)$$

$(x+i)(x+i)$

$x^2-ix+ix-i^2$

$$f(x)=k(x+3)(x^2+1)$$

$$f(x)=k(x^3+3x^2+x+3)$$

$$24=k((1)^3+3(1)^2+1+3)$$

$$24=k(8)$$

$$3=k$$

$$f(x)=3(x^3+3x^2+x+3)$$

$$\boxed{f(x)=3x^3+9x^2+3x+9}$$

Descartes's Rule of Signs

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients.

1. The number of *positive real zeros* of f is either
   a. the same as the number of sign changes of $f(x)$

   or

   b. less than the number of sign changes of $f(x)$ by a positive even integer.

   If $f(x)$ has only one variation in sign, then f has exactly one positive real zero.
2. The number of *negative real zeros* of f is either
   a. the same as the number of sign changes of $f(-x)$

   or

   b. less than the number of sign changes of $f(-x)$ by a positive even integer.

   If $f(-x)$ has only one variation in sign, then f has exactly one negative real zero.

6

4, 2, 0

3

1

The rule is based on considering *variations in sign* between consecutive coefficients. For example, the function $f(x) = 3x^7 - 2x^5 - x^4 + 7x^2 + x - 3$ has three sign changes:

$$f(x) = 3x^7 - 2x^5 - x^4 + 7x^2 + x - 3.$$

sign change sign change sign change

3

1

Descartes's Rule of Signs and Positive Real Zeros

Polynomial Function	Sign Changes	Conclusion
$f(x) = 3x^7 - 2x^5 - x^4 + 7x^2 + x - 3.$ sign change sign change sign change	3	There are 3 positive real zeros. or There is $3 - 2 = 1$ positive real zero.
$f(x) = 4x^5 + 2x^4 - 3x^2 + x + 5$ sign change sign change	2	There are 2 positive real zeros. or There are $2 - 2 = 0$ positive real zeros.
$f(x) = -7x^6 - 5x^4 + x + 9$ sign change	1	There is 1 positive real zero.

The number of real zeros given by Descartes's Rule of Signs includes rational zeros from a list of possible rational zeros, as well as irrational zeros not on the list. It does not include any imaginary zeros.

Descartes's Rule of Signs

Determine the possible numbers of positive and negative real zeros of f(x)=$x^3 + 2x^2 - 5x - 6$.

To find possibilities for positive real zeros, count the number of sign changes in the equation for f(x). There is one variation in sign change, so there is one positive real zero.

Now substitute in -x for x:

f(x)=$(-x)^3 + 2(-x)^2 - 5(-x) - 6$

f(x)= $-x^3 + 2x^2 + 5x - 6$

There are two sign changes so there are either 2 negative real zeros or none. There has to be 2 to give you a total of 3. The zeros are 2, -1, -3.

Example For $f(x)=x^3-3x^2-x+3$ how many positive and negative zeros are there? What are the zeros of the function?

Positive Real Roots

$f(x)=x^3-3x^2-x+3$

2 sign changes

$\therefore$ 2 positive real zeros or 0 positive real zeros

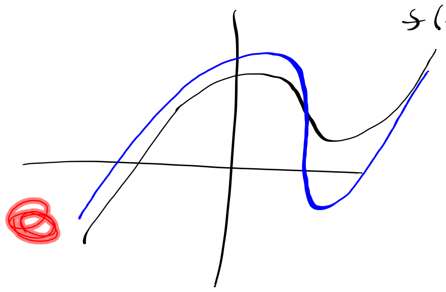
Negative Real Zeros

$f(x)=x^3-3x^2-x+3$

$f(-x)=(-x)^3-3(-x)^2-(-x)+3$

$f(-x)=-x^3-3x^2+x+3$

1 negative real zero



$f(x)=4x^4-x^3+5x^2-2x-6$

determine # of positive $\mathbb{R}$ zeros

\# of negative $\mathbb{R}$ zeros

Example

For f(x)=x3- x2+4x- 4 how many positive and negative zeros are there? Use a graphing utility to find

List all possible rational zeros of the
- 6x-8. +3x 2

$\pm\frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8$ (a)

$\pm 1, \pm 2, \pm 4, \pm 8$ (b)

(c)

$\pm\frac{1}{2}, \pm 2, \pm 4,$ (d)

$\pm 1,$

Find a third-degree polynomial function f(x) with real coefficients that have 1 and $2i$ as zeros and such that f(1)=0.

(a) $f(x) = x^3 - x^2 - 4x + 4$

(b) $f(x) = 2x^3 - x^2 + 4x - 8$

(c) $f(x) = -2x^3 + 2x^2 - 8x + 8$

(d) $f(x) = x^3 - x^2 + 4x - 4$

What are the zeros of the function f(x)=x³+2x²−8x+16? Find the first zero using a graphing utility.

(a) $-2, 2\sqrt{2}i, -2\sqrt{2}i$

(b) $2, 2\sqrt{2}, -2\sqrt{2}$

(c) $2, -2, 2i$

(d) $-2, -2, 2$