Section 2.6
Rational Functions
and
Their Graphs

Rational Functions

Rational Functions are quotients of polynomial functions. This means that rational functions can be expressed as $f(x)=\frac{p(x)}{q(x)}$ where p and q are polynomial functions and $q(x) \neq 0$. The domain of a rational function is the set of all real numbers except the x-values that make the denominator zero.

Example

x values

Find the domain of the rational function.

$$f(x)=\frac{x^2-16}{x-4}$$

$x-4 \neq 0$
$x \neq 4$

all R, $x \neq 4$

Example

Find the domain of the rational function.

$$f(x) = \frac{x}{x^2 - 36} = \frac{x}{(x-6)(x+6)}$$

all $\mathbb{R}$ $x \neq 6, x \neq -6$

Vertical Asymptotes of Rational Functions

x approaches 0 from the left.

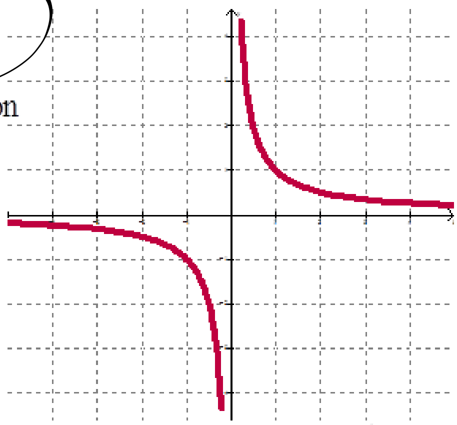
x	-1	-0.5	-0.1	-0.01	-0.001
$f(x) = \frac{1}{x}$	-1	-2	-10	-100	-1000

x approaches 0 from the right.

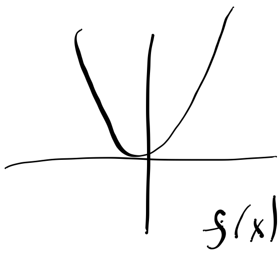
x	0.001	0.01	0.1	0.5	1
$f(x) = \frac{1}{x}$	1000	100	10	2	1

The equation $f(x)=\dfrac{1}{x}$

Vertical Asymptote on the y-axis.



$x=0$



$f(x)=x^2$

x decreases without bound:

x	-1	-10	-100	-1000
$f(x) = \frac{1}{x}$	-1	-0.1	-0.01	-0.001

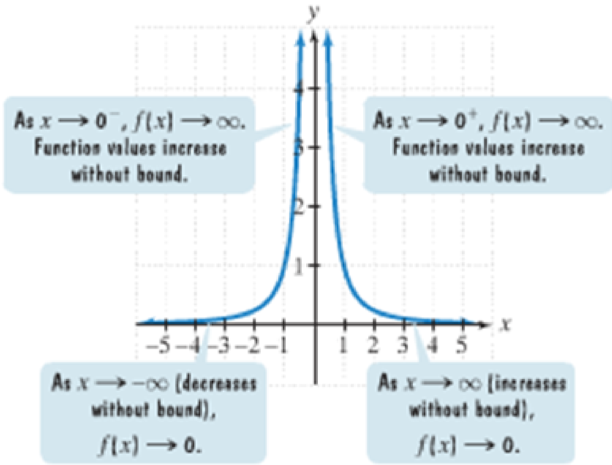
x increases without bound:

x	1	10	100	1000
$f(x) = \frac{1}{x}$	1	0.1	0.01	0.001

verticle asymptote is a...
verticle line that the
value of the function approaches
a x gets closer to the verticle line

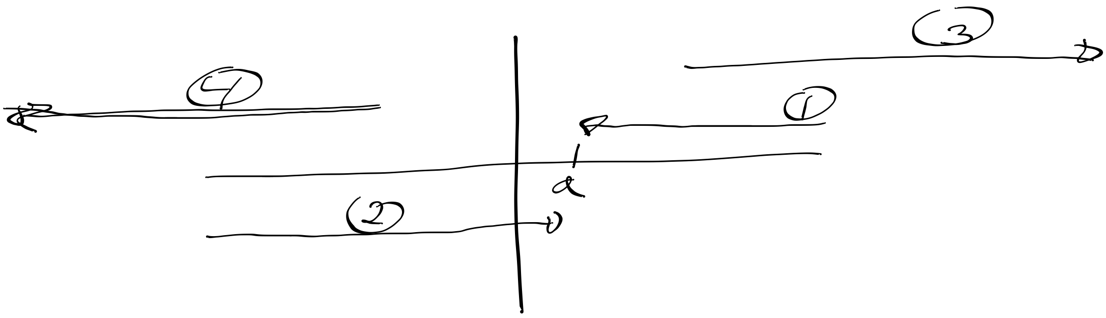
The equation $f(x)=\frac{1}{x^2}$

Vertical Asymptote on the y-axis.



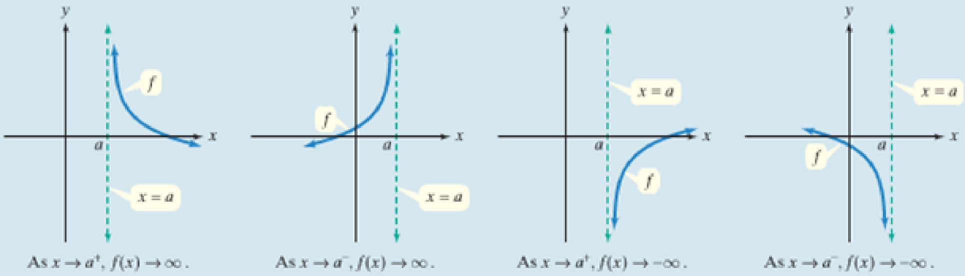
Arrow Notation

Symbol	Meaning
$x \to a^+$	x approaches a from the right.
$x \to a^-$	x approaches a from the left.
$x \to \infty$	x approaches infinity; that is, x increases without bound.
$x \to -\infty$	x approaches negative infinity; that is, x decreases without bound.



Definition of a Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a.



Thus, as x approaches a from either the left or the right, $f(x) \to \infty$ or $f(x) \to -\infty$.

Locating Vertical Asymptotes

If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function in which $p(x)$ and $q(x)$ have no common factors and a is a zero of $q(x)$, the denominator, then $x = a$ is a vertical asymptote of the graph of f.

Two Graphs with Vertical Asymptotes, one

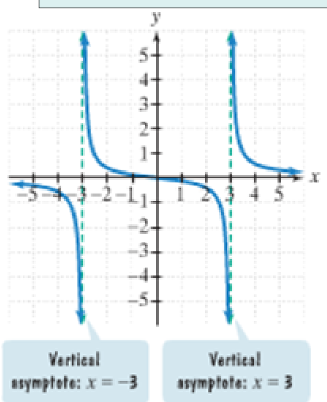


Figure 3.28(a)

The graph of $f(x) = \dfrac{x}{x^2 - 9}$ has two vertical asymptotes.

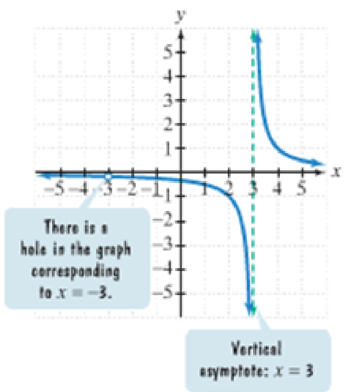


Figure 3.28(b)

The graph of $g(x) = \dfrac{x+3}{x^2 - 9}$ has one vertical asymptote.

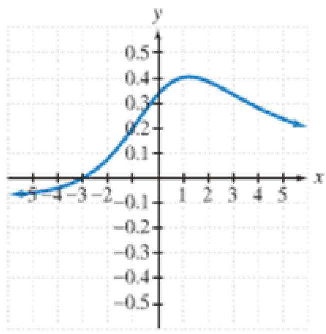


Figure 3.28(c)

The graph of $h(x) = \dfrac{x+3}{x^2 + 9}$ has no vertical asymptotes.

Example

Find the vertical asymptote, if any, of the graph of the rational function.

$$f(x) = \frac{x}{x^2 - 36}$$

Example

Find the vertical asymptote, if any, of the graph of the rational function.

$$f(x) = \frac{x}{x^2 + 36}$$

Example

Find the vertical asymptote, if any, of the graph of the rational function.

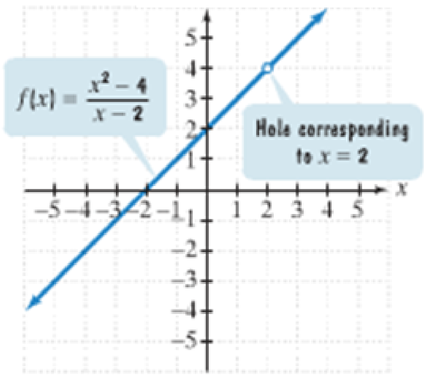
$$f(x) = \frac{x+6}{x^2-36}$$

Consider the function $f(x)=\frac{x^2-4}{x-2}$. Because the denominator is zero when x=2, the function's domain is all real numbers except 2. However, there is a reduced form of the equation in which 2 does not cause the denominator to be zero.

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = x+2,\ x \neq 2.$$

Denominator is zero at $x = 2$.

In this reduced form, 2 does not result in a zero denominator.



A graph with a hole corresponding to the denominator's zero. Your calculator will not show the hole.

Horizontal Asymptotes of Rational Functions

Definition of a Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.

Locating Horizontal Asymptotes

Let f be the rational function given by

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \quad a_n \neq 0, b_m \neq 0.$$

The degree of the numerator is n. The degree of the denominator is m.

1. If $n < m$, the x-axis, ~~or $y = 0$~~, is the horizontal asymptote of the graph of f.
2. If $n = m$, the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote of the graph of f.
3. If $n > m$, the graph of f has no horizontal asymptote.

$$f(x) = \frac{6x^4 + 3x^3 - 2x^2 + x - 6}{2x^4 + 2x^3 - 3x^2 - 3x + 16}$$

$$\frac{x^2}{x^3}$$

$$y = \frac{6}{2} = 3$$

$n < m$	$y = 0$
$n = m$	use coefficients $y = \frac{a}{b}$
$n > m$	No HA

Two Graphs with Horizontal Asymptotes, one

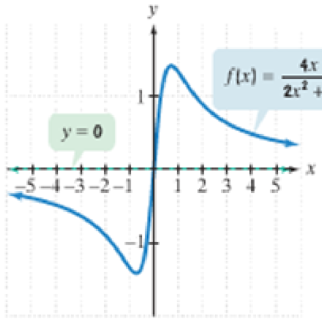


Figure 3.30(a) The horizontal asymptote of the graph is $y = 0$.

Notice how the horizontal asymptote intersects the graph.

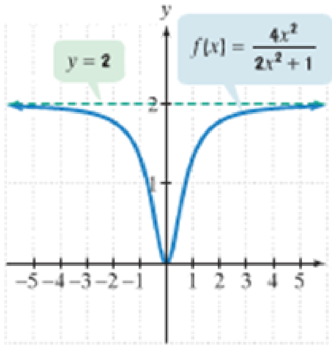


Figure 3.30(b) The horizontal asymptote of the graph is $y = 2$.

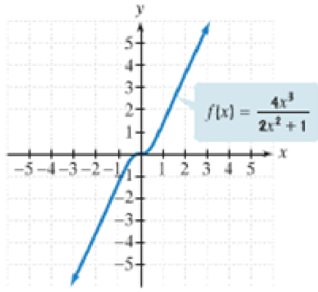


Figure 3.30(c) The graph has no horizontal asymptote.

Example **Find the horizontal asymptote, if any, of the graph of the rational function.**

$$f(x) = \frac{3x}{x^2+1}$$

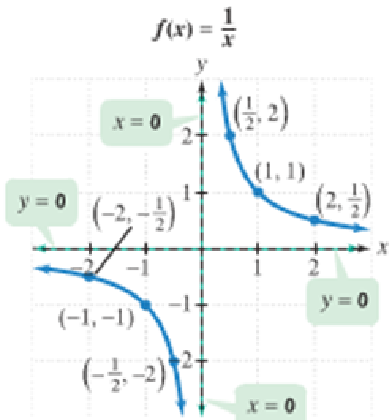
Example

Find the horizontal asymptote, if any, of the graph of the rational function.

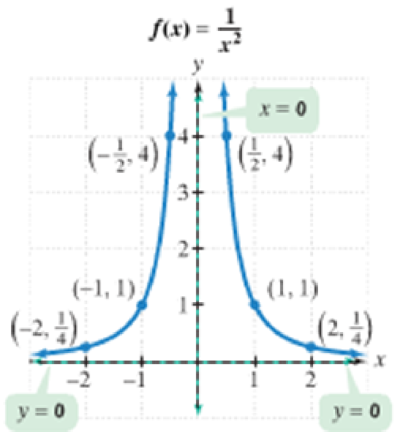
$$f(x) = \frac{6x^2}{x^2+1}$$

Using Transformations to Graph Rational Functions

Graphs of Common Rational Functions

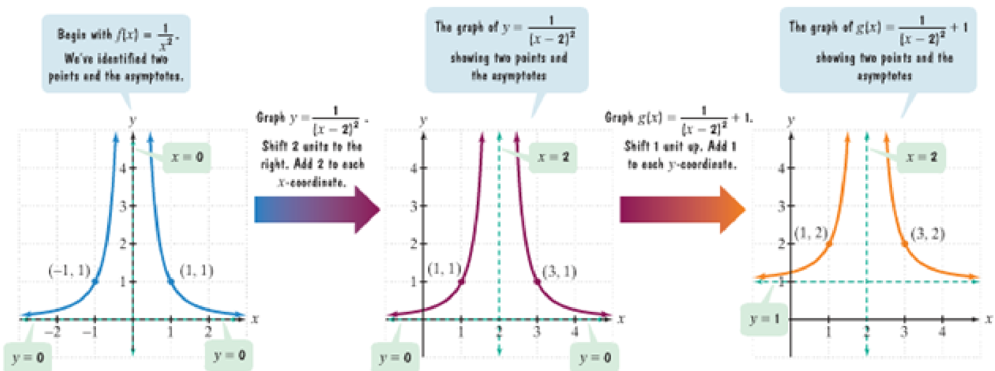


- Odd function: $f(-x) = -f(x)$
- Origin symmetry

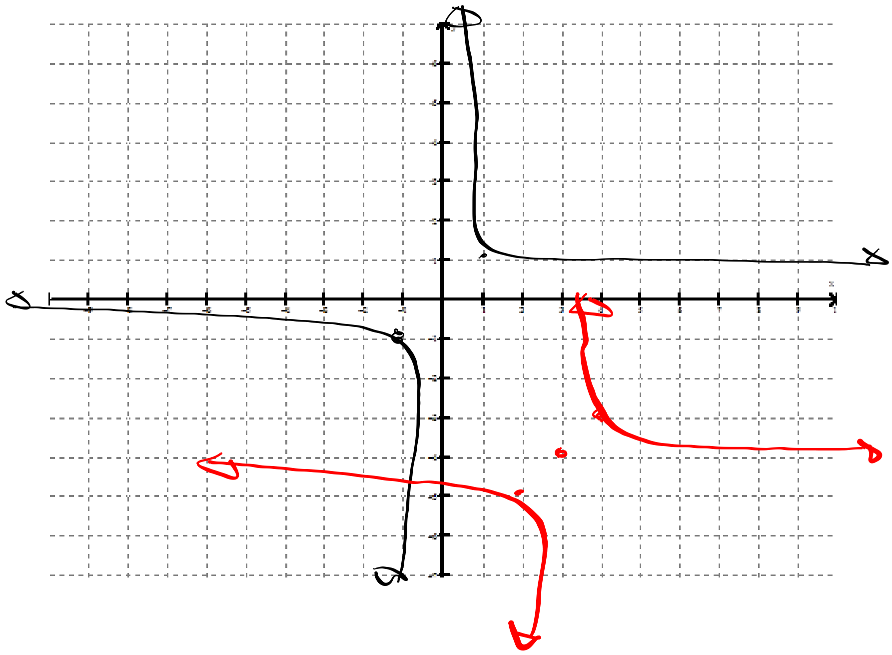


- Even function: $f(-x) = f(x)$
- y-axis symmetry

Transformations of Rational Functions

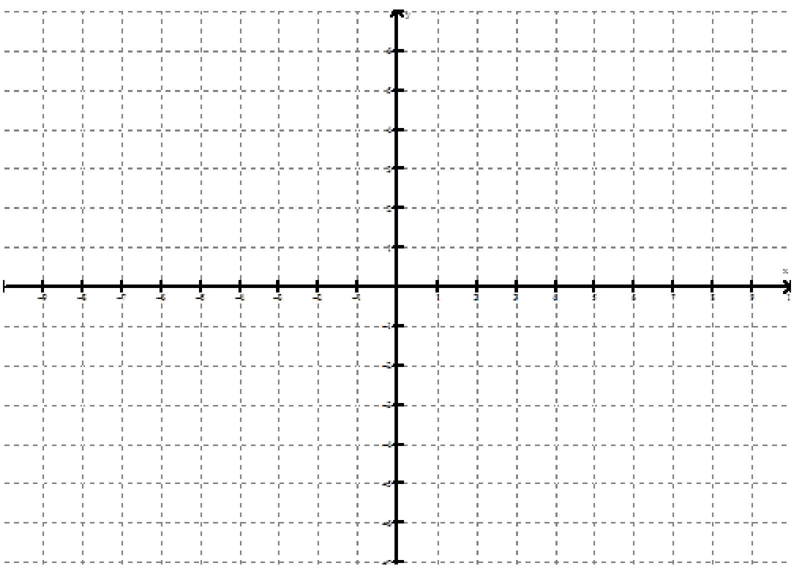


Example Use the graph of $f(x)=\frac{1}{x}$ to graph $g(x)=\frac{1}{x-3}-4$



Example

Use the graph of $f(x)=\frac{1}{x^2}$ to graph $g(x)=\frac{1}{x^2-4}+2$



Graphing Rational Functions

Strategy for Graphing a Rational Function

The following strategy can be used to graph

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions with no common factors.

1. Determine whether the graph of f has symmetry.

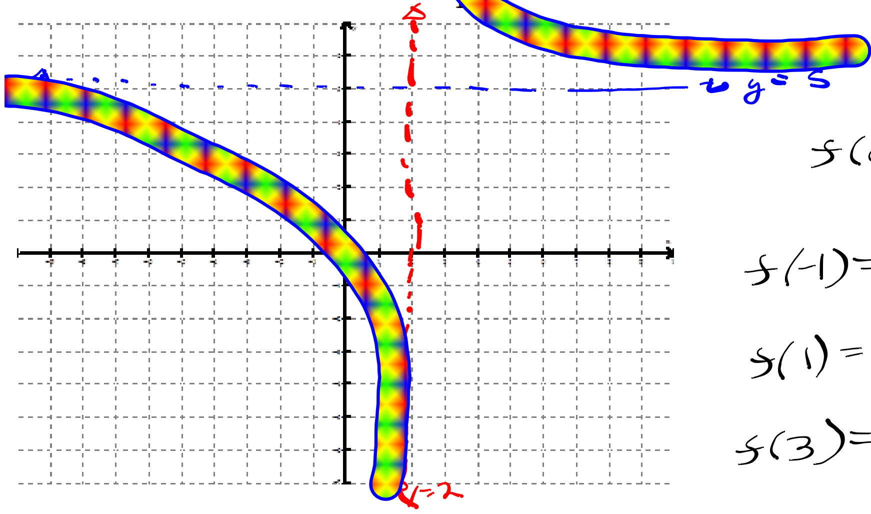
   $f(-x) = f(x)$: y-axis symmetry

   $f(-x) = -f(x)$: origin symmetry

2. Find the y-intercept (if there is one) by evaluating $f(0)$.
3. Find the x-intercepts (if there are any) by solving the equation $p(x) = 0$.
4. Find any vertical asymptote(s) by solving the equation $q(x) = 0$.
5. Find the horizontal asymptote (if there is one) using the rule for determining the horizontal asymptote of a rational function.
6. Plot at least one point between and beyond each x-intercept and vertical asymptote.
7. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.

Example

Graph $f(x)=\dfrac{5x}{x-2}$ using the 7 step strategy from the previous slide.



$\cancel{\dfrac{3x}{x-2}=0}$

$\cancel{3x=0}$

$\cancel{x=0}$

$f(0)=\dfrac{3(0)}{0-2}=0$

$f(-1)=\dfrac{3(-1)}{-1-2}=1$

$f(1)=\dfrac{3(1)}{1-2}=-3$

$f(3)=\dfrac{3(3)}{3-2}=9$

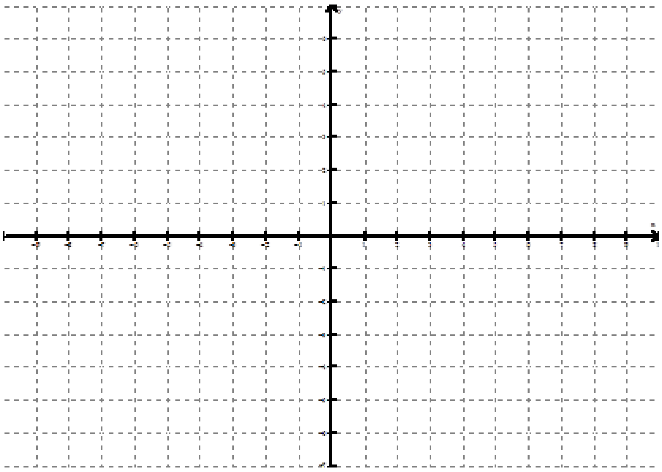
2.6

342-343

#1-69 odds

Example

Graph $f(x)=\frac{2x^2}{x^2-25}$ using the 7 step strategy.



Slant Asymptotes

The graph of a rational function has a slant asymptote if the degree of the numerator is one more than the degree of denominator. The equation of the slant asymptote can be found by division. It is the equation of the dividend with the term containing the remainder dropped.

$$\frac{p(x)}{q(x)} = \underbrace{mx + b}_{\text{Slant asymptote: } y = mx + b} + \frac{\text{remainder}}{q(x)}.$$

Example

Find the slant asymptote of the function $f(x)=\dfrac{x^2+6x+2}{x}$.

Example

Find the slant asymptote of the function f(x)= $\dfrac{x^3 - 1}{x^2 + 2x + 2}$.

Find the vertical asymptote(s) for the graph of the rational function f(x)= $\dfrac{4x^2}{x^2 - 16}$.

(a) $y = 4$
(b) $x = 4$
(c) $x = 6, -6$
(d) $y = 4, -4$

Find the horizontal asymptote(s) for the graph of the rational function $f(x)=\frac{4x^2}{x^2-16}$.

(a) $y=4$
(b) $x=4$
(c) $x=6,-6$
(d) $y=4,-4$

Find the horizontal asymptote for $f(x)=\frac{3x^3}{x^2-36}$.

(a) $y=3$
(b) $y=6,-6$
(c) $x=6,-6$
(d) $none$