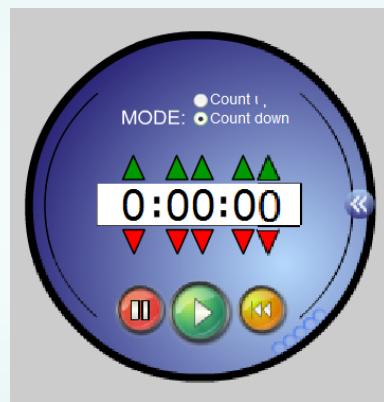


Section 5.3

Double-Angle, Power-Reducing, and Half-Angle Formulas

Question: DLIQ



$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta \\ &= 2 \sin \theta \cos \theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta \\ &= \cos^2 \theta - \sin^2 \theta\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \cdot \tan \theta} \\ &= \frac{2 \tan \theta}{1 - \tan^2 \theta}\end{aligned}$$

Double-Angle Formulas

What are the double angle formulas?

Double-Angle Formulas

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta}\end{aligned}$$

To prove each of these formulas, we replace α and β by θ in the sum formulas for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, and $\tan(\alpha + \beta)$.

- $\sin 2\theta = \sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta = 2 \sin \theta \cos \theta$

We use
 $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$

- $\cos 2\theta = \cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta = \cos^2 \theta - \sin^2 \theta$

We use
 $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$

- $\tan 2\theta = \tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

We use
 $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$

What are some common errors that Mr. Wee does not want to see?

Study Tip

The 2 that appears in each of the double-angle expressions cannot be pulled to the front and written as a coefficient.

Incorrect!

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \\ \cos 2\theta &= 2 \cos \theta \\ \tan 2\theta &= 2 \tan \theta\end{aligned}$$



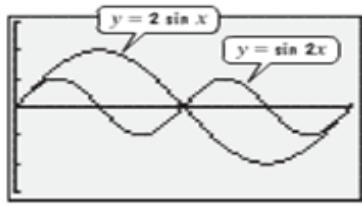
The figure shows that the graphs of

$$y = \sin 2x$$

and

$$y = 2 \sin x$$

do not coincide: $\sin 2x \neq 2 \sin x$.

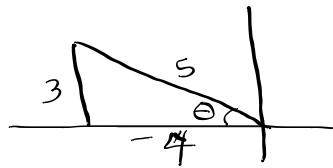


$[0, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

Three Forms of the Double-Angle Formula for $\cos 2\theta$

$$\begin{aligned}\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \cos 2\theta &= 2 \cos^2 \theta - 1 \\ \cos 2\theta &= 1 - 2 \sin^2 \theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= \cos^2 \theta - (1 - \cos^2 \theta) \\ &= 2 \cos^2 \theta - 1\end{aligned}$$



$$\begin{aligned} 3^2 + x^2 &= 5^2 \\ 9 + x^2 &= 25 \\ x^2 &= 16 \\ x &= 4 \end{aligned}$$

$\frac{\pi}{2}$
I
II
III
IV

Example

If $\sin \theta = \frac{3}{5}$ and θ lies in Quadrant II, find the exact value of each of the following.

$$\begin{aligned} a. \sin 2\theta &= 2 \sin \theta \cos \theta \\ &= 2 \cdot \frac{3}{5} \cdot -\frac{4}{5} = \left(\frac{-24}{25} \right) \end{aligned}$$

$$b. \cos 2\theta = \frac{\cos^2 \theta - \sin^2 \theta}{(-\frac{4}{5})^2 - (\frac{3}{5})^2} = \frac{\frac{16}{25} - \frac{9}{25}}{\frac{16}{25}} = \left(\frac{7}{25} \right)$$

$$c. \tan 2\theta = \frac{2 \cdot \frac{3}{4}}{1 - (-\frac{3}{4})^2} = \frac{\frac{-6}{4}}{\frac{16}{16} - \frac{9}{16}} = -\frac{6}{4} \cdot \frac{16}{7} = \left(\frac{-24}{7} \right)$$

Example

(12)

Verify the identity: $\sin 4x = 4 \sin x \cos^3 x - 4 \sin^3 x \cos x$

Break $4x$ into $2x+2x$

$\sin(2x+2x)$

sum identity

$$\sin 2x \cos 2x + \cos 2x \sin 2x$$

double angle identity

$$2 \sin x \cos x (\cos^2 x - \sin^2 x) + (\cos^2 x - \sin^2 x) \cdot 2 \sin x \cos x$$

factor out the
 $\cos^2 x - \sin^2 x$

$$(2 \sin x \cos x + 2 \sin x \cos x)(\cos^2 x - \sin^2 x)$$

Simplify

$$(4 \sin x \cos x)(\cos^2 x - \sin^2 x)$$

Distribute

$$4 \sin x \cos^3 x - 4 \sin^3 x \cos x$$

Example

Verify the identity: $\cos 3x = 4 \cos^3 x - 3 \cos x$

Example

~~Verify the identity: $\cot x - 2 \cot 2x = \tan x$~~

What is the double angle identity for $\cot 2\theta$?

Start with the identity for $\tan 2\theta$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\frac{1}{2} = \frac{2}{4}$$

$$\frac{2}{1} = \frac{4}{2}$$

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{1 - \tan^2 \theta}{2 \tan \theta}$$

$$\cot 2\theta = \frac{1 - \tan^2 \theta}{2 \tan \theta}$$

Example

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \tan 2\theta$$

Find the exact value of $\frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}} = \tan 2 \cdot \frac{\pi}{8}$

$$= \tan \frac{2\pi}{8}$$

$$= \tan \frac{\pi}{4}$$

$\textcircled{=} 1$

Example

Find the exact value of $\frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ}$

Power-Reducing Formulas

What are the power reducing identities?

Power-Reducing Formulas

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \tan^2 \theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

Both sin and cos have cos in the numerator...

$\tan^2 x$ is just the quotient property applied

Example

Write an equivalent expression for $\sin^4 x$ that does not contain powers of trigonometric functions greater than 1.

$$\begin{aligned}
 & \sin^4 x && \text{given} \\
 & (\sin^2 x)^2 && \text{rewrite} \\
 & \left(\frac{1-\cos 2x}{2}\right)^2 && \text{power reduction for sin} \\
 & \frac{1-2\cos 2x + \cos^2 2x}{4} && \text{expanding... careful with the numerator... binomial} \\
 & \frac{1-2\cos 2x + \frac{1+\cos 2(2x)}{2}}{4} && \text{power reduction for cos... careful... theta=2x here...} \\
 & \frac{2-4\cos 2x + 1+\cos 4x}{8} && \text{simplify} \\
 & \frac{3-4\cos 2x + \cos 4x}{8} && \text{simplify}
 \end{aligned}$$

Half-Angle Formulas

What are the half-angle formulas?

Half-Angle Formulas

() $\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$

() $\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$

() $\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$

The \pm symbol in each function. I circled it in red.

Half-Angle Formulas for Tangent

$$\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}$$

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$$

lues for
1 of the
 α lies

Study Tip

The $\frac{1}{2}$ that appears in each of the half-angle formulas cannot be pulled to the front and written as a coefficient.

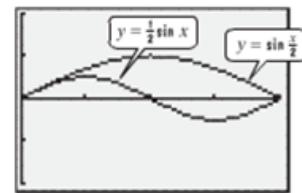
Incorrect!

$$\sin \frac{\theta}{2} = \frac{1}{2} \sin \theta$$

$$\cos \frac{\theta}{2} = \frac{1}{2} \cos \theta$$

$$\tan \frac{\theta}{2} = \frac{1}{2} \tan \theta$$

The figure shows that the graphs of $y = \sin \frac{x}{2}$ and $y = \frac{1}{2} \sin x$ do not coincide: $\sin \frac{x}{2} \neq \frac{1}{2} \sin x$.



$[0, 2\pi, \frac{\pi}{2}]$ by $[-2, 2, 1]$

Study Tip

Keep in mind as you work with the half-angle formulas that the sign *outside* the radical is determined by the half angle $\frac{\alpha}{2}$. By contrast, the sign of $\cos \alpha$, which appears *under* the radical, is determined by the full angle α .

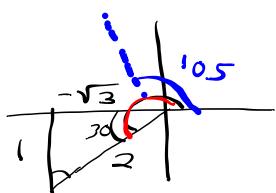
$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

The sign is determined
by the quadrant of $\frac{\alpha}{2}$.

The sign of $\cos \alpha$
is determined by
the quadrant of α .

Example

Find $\sin 105^\circ$. ~~use $105^\circ = \frac{1}{2}(210^\circ)$~~ .



$$\begin{aligned}\sin \frac{210}{2} &= \pm \sqrt{\frac{1 - \cos 210}{2}} \\ &= \pm \sqrt{\frac{1 - (-\frac{\sqrt{3}}{2})}{2}} \\ &= \pm \sqrt{\frac{\frac{2 + \sqrt{3}}{2}}{2}}\end{aligned}$$

$$= \boxed{\sqrt{\frac{2 + \sqrt{3}}{4}}}$$

How do I figure the sign...???

Draw a picture... do not confuse your tool-angle with your actual angle...

Example

Find the $\cos \frac{5\pi}{8}$. Use $\frac{1}{2} \left(\frac{5\pi}{4} \right)$.

Example

Find the $\sin 75^\circ$ using the $\sin \frac{x}{2}$ formula.

Example

Find the $\cos 105^\circ$ using the $\cos \frac{x}{2}$ formula.

Example

Find the $\tan \frac{5\pi}{8}$. Use $\frac{1}{2} \left(\frac{5\pi}{4} \right)$.

Example

Find the $\tan 105^\circ$ using the $\tan \frac{x}{2}$ formula.

Principal Trigonometric Identities**Sum and Difference Formulas**

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \quad \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

Double-Angle Formulas

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Power-Reducing Formulas

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \tan^2 \theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

Half-Angle Formulas

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$$

Homework:

Assignment 5.3: pp597: 1-67 odds...

- 1.) Do the HW
- 2.) Grade the problem that are in the back
- 3.) Write this score next to your name on the first page...

10 homework extra for evens... separate

Due Tuesday

Project:

Make simple flashcards for the trig identities...

Due Tuesday

If $\tan \theta = -1$ and θ lies in quadrant II, find $\cos 2\theta$.

1 (a)

-1 (b)

0 (c)

undefined (d)



If the $\tan \alpha = \frac{7}{24}$, find the $\tan \frac{\alpha}{2}$.

- (a) $\frac{7}{48}$
- (b) $\frac{7}{12}$
- (c) $\frac{3}{4}$
- (d) $\frac{1}{7}$

