Objective and why: "un-add" rational expressions. We need this in calculus to integrate certain functions... I'll show you one at the end of class

Topic: 7.3: Partial Fractions

Question: What is a partial fraction? What are the four cases of partial fraction decompositions that we covered? DLIQ

Homework: page 745: 1-8 all, 9-41 odds
(Due Thursday)

Reminders: Due today: 7.2; Missing Work; TEST CORRECTIONS

MODE: Count up / Count down
0:00:00
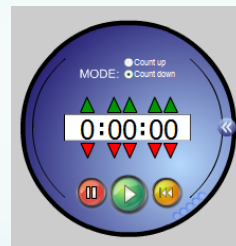
1.) $A + B = 6$ $\xrightarrow{\times 3}$ $3A + 3B = 18$

$-5A - 3B = 14$ $\xrightarrow{\times 1}$ $-5A - 3B = 14$

$$-2A = 32$$
$$A = -16$$

$A + B = 6$
$-16 + B = 6$
$B = 22$

$(-16, 22)$

$$\frac{2}{x} \quad + \quad \frac{-2}{x+2} \quad + \quad \frac{-3}{(x+2)^2}$$

$$\frac{2}{x} \quad + \quad \frac{-2}{x+2} \quad + \quad \frac{-3}{(x+2)^2}$$

$$\frac{x(x+2)^2}{}$$

$$\frac{2(x+2)^2}{x(x+2)^2} \quad + \quad \frac{-2(x(x+2))}{x(x+2)^2} \quad + \quad \frac{-3x}{x(x+2)^2}$$

$$\frac{2x^2+8x+8 \; + \; -2x^2-4x \; -3x}{x(x+2)^2}$$

adding rational expressions

← ————————————————————————— •

$$\frac{x+8}{x(x+2)^2} \quad = \quad \frac{2}{x} \quad + \quad \frac{-2}{x+2} \quad + \quad \frac{-3}{(x+2)^2}$$

partial
fraction   partial
fraction   partial
fraction

• ————————————————————————— →
finding the partial fraction decomposition, "unadding"

The Idea Behind Partial Fraction Decomposition

$$\frac{x + 14}{(x - 4)(x + 2)} \text{ is expressed as the sum of two simpler fractions.}$$

Partial fraction   **Partial fraction**

$$\frac{x + 14}{(x - 4)(x + 2)} = \frac{3}{x + 4} + \frac{-2}{x + 2}.$$

This is the partial fraction decomposition of $\frac{x + 14}{(x - 4)(x + 2)}$.

Each of the two fractions on the right is called a partial fraction. The sum of these fractions is called the partial fraction decomposition of the rational expression on the left-hand side.

$$\frac{9x^2 - 9x + 6}{(2x - 1)(x + 2)(x - 2)}$$

$P(x) = 9x^2 - 9x + 6$; highest power = 2

$Q(x) = (2x - 1)(x + 2)(x - 2)$; multiplying factors, highest power = 3.

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2}.$$

$P(x) = 5x^3 - 3x^2 + 7x - 3$; highest power = 3

$Q(x) = (x^2 + 1)^2$; squaring the expression, highest power = 4.

The partial fraction decomposition of a rational expression depends on the factors of the denominator. We consider four cases involving different kinds of factors in the denominator:

1. The denominator is a product of distinct linear factors.

2. The denominator is a product of linear factors, some of which are repeated.

3. The denominator has prime quadratic factors, none of which is repeated.

4. The denominator has a repeated prime quadratic factor.
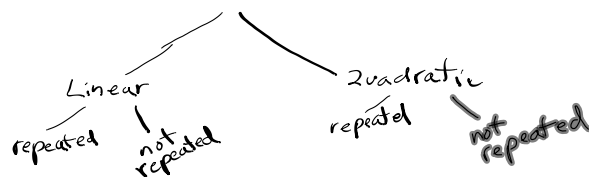
Linear

repeated   not repeated

Quadratic

repeated   not repeated

The Partial Fraction Decomposition of a Rational Expression with Distinct Linear Factors in the Denominator

$$\frac{A}{ax + b}.$$

Constant

Linear factor

Each distinct linear factor in the denominator produces a partial fraction of the form *constant over linear factor*. For example,

$$\frac{9x^2 - 9x + 6}{(2x - 1)(x + 2)(x - 2)} = \frac{A}{2x - 1} + \frac{B}{x + 2} + \frac{C}{x - 2}.$$

We write a constant over each linear factor in the denominator.

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has Distinct Linear Factors

The form of the partial fraction decomposition for a rational expression with distinct linear factors in the denominator is

$$\frac{P(x)}{(a_1x + b_1)(a_2x + b_2)(a_3x + b_3)\cdots(a_nx + b_n)}$$

$$= \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \frac{A_3}{a_3x + b_3} + \cdots + \frac{A_n}{a_nx + b_n}.$$

How do I set-up a partial fraction decomposition (pfd) for distinct linear denominators?

Each linear factor (in the denominator) produces one partial fraction with a constant as the numerator...

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Steps in Partial Fraction Decomposition

1. Set up the partial fraction decomposition with the unknown constants A, B, C, etc., in the numerator of the decomposition.
2. Multiply both sides of the resulting equation by the least common denominator.
3. Simplify the right-hand side of the equation.
4. Write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms.
5. Solve the resulting linear system for A, B, C, etc.
6. Substitute the values for A, B, C, etc., into the equation in step 1 and write the partial fraction decomposition.

Study Tip

You will encounter some examples in which the denominator of the given rational expression is not already factored. If necessary, begin by factoring the denominator. Then apply the six steps needed to obtain the partial fraction decomposition.

Example   Find the partial fraction decomposition of

$$\frac{6x+14}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

$$\frac{(x-3)(x+5)}{1}\left(\frac{6x+14}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5} \right)$$

$$6x+14 = A(x+5) + B(x-3)$$

$$6x+14 = Ax + 5A + Bx - 3B$$

$$6x+14 = Ax + Bx + 5A - 3B$$

$$6x+14 = (A+B)x + 5A - 3B$$

$$A+B = 6 \quad \xrightarrow{\times 3} \quad 3A+3B = 18$$
$$5A-3B = 14 \quad \xrightarrow{\times 1} \quad \underline{5A-3B = 14}$$
$$8A = 32$$
$$A = 4$$
$$B = 2$$

$$\frac{6x+14}{(x-3)(x+5)} = \frac{4}{x-3} + \frac{2}{x+5}$$

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The Partial Fraction Decomposition of a Rational Expression with Linear Factors in the Denominator, Some of Which are Repeated

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has Repeated Linear Factors

The form of the partial fraction decomposition for a rational expression containing the linear factor $ax + b$ occuring n times as its denominator is

$$\frac{P(x)}{(ax + b)^n} = \frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_n}{(ax + b)^n}.$$

Include one fraction with a constant numerator for each power of $ax + b$.

$$\frac{1}{(x+3)^4} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4}$$

How do I set-up PFD for repeated linear factors?

The repeated linear factors produce partial fractions up to the power of the repetition.

What if I have some repeated, some not repeated?

Deal with the non-repeated as before, and deal with the repeated as we just said...

$$\frac{x}{(x-1)(x+4)^3} = \frac{A}{x-1} + \frac{B}{x+4} + \frac{C}{(x+4)^2} + \frac{D}{(x+4)^3}$$

Study Tip

Avoid this common error:

INCORRECT!

$$\frac{x-18}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-3}$$

Listing $x - 3$ twice does not take into account $(x - 3)^2$.

$$1 - 5 \text{ all}$$
$$9 - 27 \text{ odds}$$

$$x^3 - x$$
$$x(x^2 - 1)$$
$$x(x+1)(x-1)$$

Example   Find the partial fraction decomposition

$$\frac{x+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\frac{x(x+2)^2}{1}\left(\frac{x+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \right)$$

$$x + 8 = A(x+2)^2 + Bx(x+2) + Cx$$

$$x + 8 = Ax^2 + 4Ax + 4A + Bx^2 + 2Bx + Cx$$

$$x + 8 = Ax^2 + Bx^2 + 4Ax + 2Bx + Cx + 4A$$

$$0x^2 + 1x + 8 = (A+B)x^2 + (4A+2B+C)x + 4A$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad B = -2$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad A = 2$$

$$A + B = 0$$

$$4A + 2B + C = 1 \quad\quad\quad 4(2) + 2(-2) + C = 1$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 4 + C = 1$$
$$4A = 8 \quad\quad\quad\quad\quad\quad\quad\quad\quad C = -3$$

$$\frac{x+8}{x(x+2)^2} = \boxed{\frac{2}{x} + \frac{-2}{x+2} + \frac{-3}{(x+2)^2}}$$

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The Partial Fraction Decomposition of a Rational Expression with Prime, Nonrepeated Quadratic Factors in the Denominator

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has a Nonrepeated, Prime Quadratic Factor

If $ax^2 + bx + c$ is a prime quadratic factor of $Q(x)$, the partial fraction decomposition will contain a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

Linear numerator

Quadratic factor

Please continue you notes on the same paper...

$$\frac{3x^2 + 17x + 14}{(x - 2)(x^2 + 2x + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 2x + 4}$$

We write a constant over the linear factor in the denominator.

We write a linear numerator over the prime quadratic factor in the denominator.

How do I set up my partial fraction if the denominator has a prime quadratic factor?

Prime quadratic factors produce partial fractions with linear numerators...

Graphing

You can use the TABLE feature of a graphing utility to check a partial fraction decomposition. To check the result of Example 3, enter the given rational function and its partial fraction decomposition:

$$y_1 = \frac{3x^2 + 17x + 14}{(x - 2)(x^2 + 2x + 4)}$$

$$y_2 = \frac{5}{x - 2} + \frac{-2x + 3}{x^2 + 2x + 4}.$$

X	Y₁	Y₂
-3	.28571	.28571
-2	.5	.5
-1	0	0
0	-1.75	-1.75
1	-4.857	-4.857
2	ERROR	ERROR
3	4.8421	4.8421

X= -3

No matter how far up or down we scroll, $y_1 = y_2$, so the decomposition appears to be correct.

Example

Find the partial fraction decomposition of

$$\left(\frac{3x^2 - 4x + 2}{(x-2)(x^2 - 3x + 5)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 3x + 5} \right) \frac{(x-2)(x^2 - 3x + 5)}{1}$$

$$3x^2 - 4x + 2 = A(x^2 - 3x + 5) + (Bx + C)(x - 2)$$

$$3x^2 - 4x + 2 = Ax^2 - 3Ax + 5A + Bx^2 - 2Bx + Cx - 2C$$

$$3x^2 - 4x + 2 = Ax^2 + Bx^2 - 3Ax - 2Bx + Cx + 5A - 2C$$

$$3x^2 - 4x + 2 = (A + B)x^2 + (-3A - 2B + C)x + 5A - 2C$$

$$A + B = 3 \quad \xrightarrow{\times 2} \quad 2A + 2B = 6$$

$$-3A - 2B + C = -4 \quad \xrightarrow{\times} \quad -3A - 2B + C = -4$$

$$\underline{ -A + C = 2}$$

$$5A - 2C = 2$$

$$-2A + 2C = 4$$

$$\overline{3A = 6}$$

$$A = 2$$

$$B = 1$$

$$C = 4$$

$$= \frac{2}{x-2} + \frac{x + 4}{x^2 - 3x + 5}$$

The Partial Fraction Decomposition of a Rational Expression with a Prime, Repeated Quadratic Factor in the Denominator

The Partial Fraction Decomposition of $\dfrac{P(x)}{Q(x)}$: $Q(x)$ Has a Prime, Repeated Quadratic Factor

The form of the partial fraction decomposition for a rational expression containing the prime factor $ax^2 + bx + c$ occuring n times as its denominator is

$$\frac{P(x)}{(ax^2 + bx + c)^n} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3x + B_3}{(ax^2 + bx + c)^3} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.$$

Include one fraction with a linear numerator for each power of $ax^2 + bx + c$.

Study Tip

When the denominator of a rational expression contains a **power of a linear factor**, set up the partial fraction decomposition with **constant numerators** (A, B, C, etc.). When the denominator of a rational expression contains a **power of a prime quadratic factor**, set up the partial fraction decomposition with **linear numerators** ($Ax + B, Cx + D$, etc.).

Study Tip

When a rational expression contains a power of a factor in the denominator, be sure to set up the partial fraction decomposition to allow for every natural-number power of that factor less than or equal to the power. Example:

$$\frac{2x + 1}{(x - 5)^2 x^3}$$

$$= \frac{A}{x - 5} + \frac{B}{(x - 5)^2} + \frac{C}{x} + \frac{D}{x^2} + \frac{E}{x^3}$$

Although $(x - 5)^2$ and x^2 are quadratic, they are still expressed as powers of linear factors, $x - 5$ and x. Thus, the numerator is constant.

Example   Find the partial fraction decomposition of

$$\cancel{\frac{2}{0}}\left(\frac{2x^3 - x - 1}{(x^2 - 2)^2} = \frac{Ax+B}{x^2-2} + \frac{Cx+D}{(x^2-2)^2} \right)\frac{(x^2-2)^2}{1}$$

$$2x^3 - x - 1 = (Ax + B)(x^2 - 2) + Cx + D$$

$$2x^3 + 0x^2 - x - 1 = Ax^3 + Bx^2 - 2Ax + Cx - 2B + D$$

$$A = 2 \qquad\qquad C = 3$$
$$B = 0 \qquad\qquad D = -1$$
$$-2A + C = -1$$
$$-2B + D = -1$$

$$= \frac{2x}{x^2-2} + \frac{3x-1}{(x^2-2)^2}$$

Find the partial fraction decomposition of

$$\frac{x-14}{(x-2)(x-5)}$$

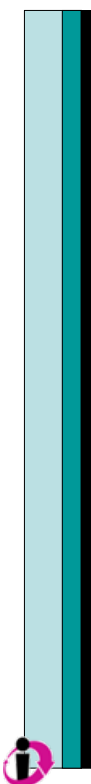(a) $\dfrac{4}{x-2} - \dfrac{3}{x-5}$

(b) $\dfrac{2}{x-2} + \dfrac{3}{x-5}$

(c) $\dfrac{1}{x-2} - \dfrac{3}{x-5}$

(d) $\dfrac{4}{x-2} + \dfrac{2}{x-5}$

Find the partial fraction decomposition of

$$\frac{2x+11}{(x+3)^2}$$

(a) $\dfrac{5}{(x+3)} + \dfrac{2}{(x+3)^2}$

(b) $\dfrac{3}{(x+3)} + \dfrac{2}{(x+3)^2}$

(c) $\dfrac{5}{(x+3)} + \dfrac{4}{(x+3)^2}$

(d) $\dfrac{2}{(x+3)} + \dfrac{5}{(x+3)^2}$

Find the partial fraction decomposition of

$$\frac{2x+11}{(x+3)^2}$$

(a) $\dfrac{5}{(x+3)} + \dfrac{2}{(x+3)^2}$

(b) $\dfrac{3}{(x+3)} + \dfrac{2}{(x+3)^2}$

(c) $\dfrac{5}{(x+3)} + \dfrac{4}{(x+3)^2}$

(d) $\dfrac{2}{(x+3)} + \dfrac{5}{(x+3)^2}$