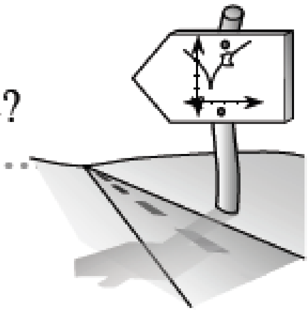
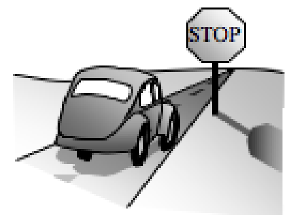
2.2.3 How do limits and continuity relate?

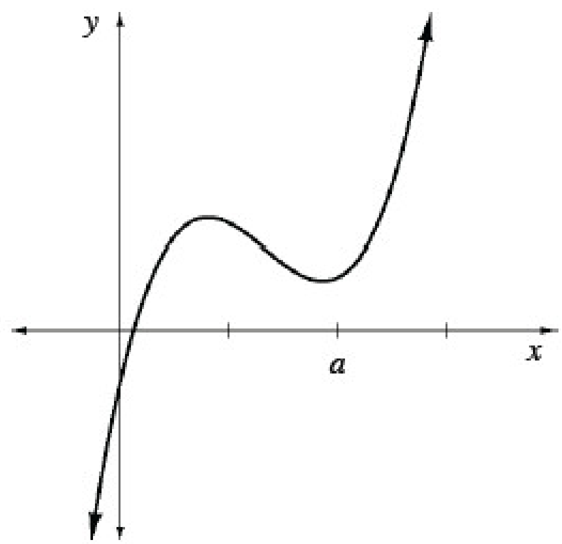


Definition of Continuity

MATH NOTES



The 3 Conditions of Continuity



Continuity is an important concept in calculus because many important theorems of calculus require continuity to be true. Simply stating that you can trace a graph without lifting your pencil is neither a complete nor a formal way to justify the continuity of a function at a point.

In order to justify that a function $f(x)$ is **continuous** at the point $x = a$, you must demonstrate that $f(x)$ meets all three conditions listed below.

1. $\lim\limits_{x \to a} f(x)$ exists

   Recall that this means: $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

2. $f(a)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

A function is **continuous on an interval** if it is continuous at each point in the interval.

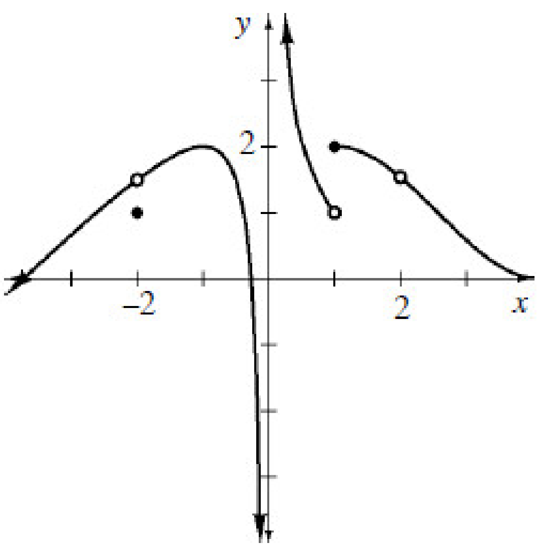
2-73. Examine the conditions of continuity given in the Math Notes box above and summarize them with your team. Then demonstrate your understanding of continuity by sketching functions for parts (a) − (c).

a. Sketch a function that satisfies condition 1, but not 2 (and therefore not 3).

b. Sketch a function that satisfies condition 2, but not 1 or 3.

c. Sketch a function that satisfies conditions 1 and 2, but not 3.

2-74. Identify which of the three conditions failed for *each* of the four discontinuities in the graph below. Notice that there are different types of discontinuities. Classify the discontinuities in a way that makes sense to you and your team.



2-75. If $g(x)$ is continuous for all real numbers, such that $g(-4) = -10$ and $g(-1) = 3$, explain why $g(x)$ *must* have a root (x-intercept) for an x-value in the interval $(-4, -1)$. Include a sketch of a possible function for $g(x)$.

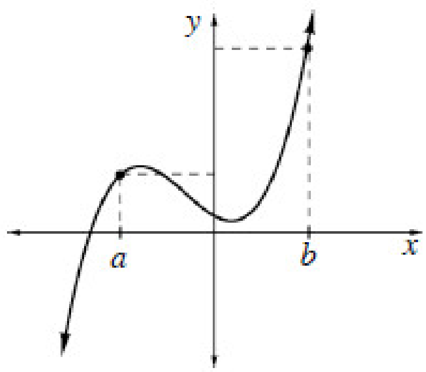
2-76. Use the 3 conditions of continuity to justify why $f(x) = |x|$ is continuous at $x = 0$.

2-77. While waiting for a bus, you and your friends see a car traveling at 65 mph. When the driver notices you, he instantly slams on the brakes and comes to a stop.

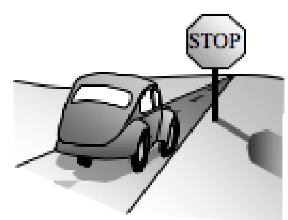
a. T/F: While breaking, the driver traveled at every *intermediate* speed between 65 mph and 0 mph.

b. Must the graph of this situation be a continuous function?

2-78. Explain why a function that is continuous for all x-values on $[a, b]$ must pass through every y-value between $f(a)$ and $f(b)$ *at least once* in that interval. This is called the **Intermediate Value Theorem** for continuous functions.



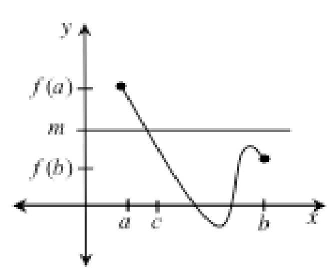
MATH NOTES



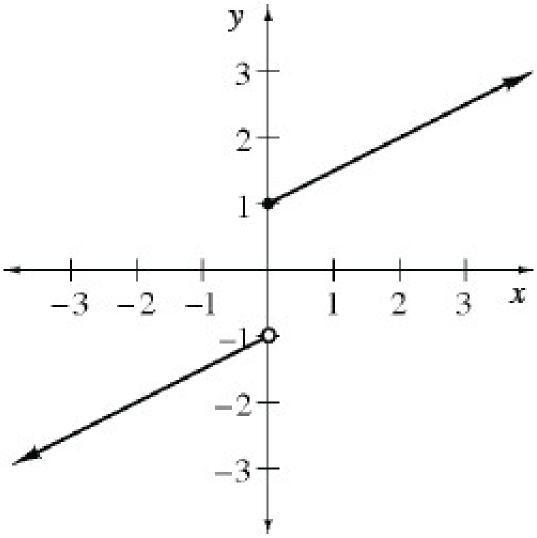
Intermediate Value Theorem

Let $f(x)$ be a function continuous on the closed interval $[a, b]$.

Then for every number m between $f(a)$ and $f(b)$ there exists a number c, $a < c < b$, such that $f(c) = m$.



2-79. Examine the function shown below. Notice that $f(-2) = -2$ and $f(2) = 2$, yet there is no root between $x = -2$ and 2. Why does this not contradict the Intermediate Value Theorem?



2-80. For some continuous function f, $f(-3) = 5$ and $f(2) = -3$. What is the minimum number of values possible for a that satisfy $f(a) = 1$?

Review & Preview

2-81. A helium balloon is released from the ground and floats upward. The height of the balloon is shown at the following times: Help (Html5)⇔Help (Java)

Time(sec)	0	1	2	3	4	5	6	7	8	9	10
Height(feet)	0	50	98	144	188	230	270	308	344	378	410

a. What was the average velocity over the first 10 seconds of the balloon's flight? Over the first 5 seconds?

b. Find the finite differences for the heights. How is the velocity changing?

c. What do the finite differences tell you about the height function for the balloon?

2-82. Examine the expanded sums below and write the equivalent sigma notation. Help (Html5)⇔Help (Java)

a. $\frac{2}{3}f(-2+\frac{2}{3}\cdot 0)+\frac{2}{3}f(-2+\frac{2}{3}\cdot 1)+\frac{2}{3}f(-2+\frac{2}{3}\cdot 2)+\frac{2}{3}f(-2+\frac{2}{3}\cdot 3)$

b. $\frac{1}{2} f(6+\frac{1}{2}\cdot 0)+\frac{1}{2} f(6+\frac{1}{2}\cdot 1)+\frac{1}{2} f(6+\frac{1}{2}\cdot 2)+\frac{1}{2} f(6+\frac{1}{2}\cdot 3)+\frac{1}{2} f(6+\frac{1}{2}\cdot 4)$

2-83. The **Intermediate Value Theorem** is sometimes used to prove that roots exist. For example, $f(x)=5\sqrt[3]{x-2}-4$ is a continuous function. Given $f(2) = -4$ and $f(3) = 1$, does $f(x)$ have a root somewhere between $x = 2$ and $x = 3$? Why or why not? Help (Html5)⇔Help (Java)

2-84. Write a Riemann sum for a general function $f(x)$ to estimate $A(f, 2 \le x \le 5)$ using n left endpoint rectangles of equal width if: Help (Html5)⇔Help (Java)

a. $n = 3$ rectangles

b. $n = 9$ rectangles

c. $n = 300$ rectangles

2-85. Jamal wrote the following Riemann sum to estimate the area under $f(x) = 3x^2 - 2$. Help (Html5)⇔Help (Java)

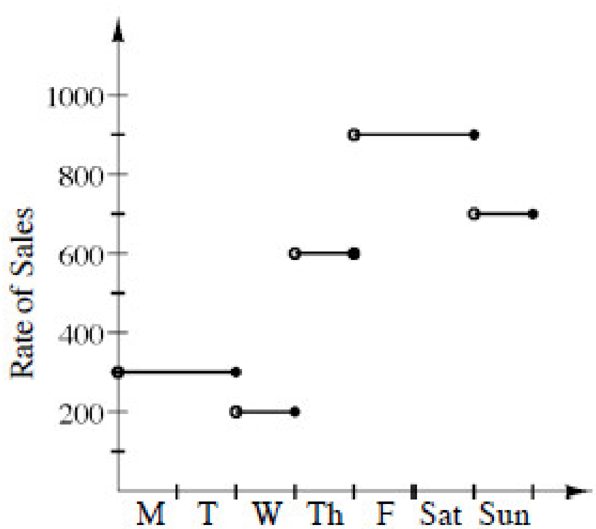
$$\sum_{i=0}^{9} \frac{1}{2} f\left(-3+\frac{1}{2} i\right)$$

a. Draw a sketch of the region. How many rectangles did he use?

b. For what domain of $f(x)$ did Jamal estimate the area?

c. Use the summation feature of your calculator to find the approximate area using Jamal's Riemann sum.

2-86. The manager of Books-To-Go knows that the rate of daily sales (in books per day) varies over the course of a week. This rate can be represented by the step function shown in the graph below. Using this data, calculate how many books this store sold during this week. What is the average number of books sold per day? Help (Html5)⇔Help (Java)



2-87. For each description below, write a limit equation and sketch a possible function. Help (Html5)⇔Help (Java)

a. As $x \to 0$, $y \to 9$.

b. As x gets closer to 3 on both sides, $f(x)$ becomes increasingly large.

c. Which of the limits from parts (a) and (b) exist? Explain your reasoning.

2-88. Given $f(x)=\begin{cases} x^2 & \text{for } x<1, \text{but } x\neq -1 \\ 3 & \text{for } x=1 \\ 2x-1 & \text{for } 1<x<3 \\ 4 & \text{for } x\geq 3 \end{cases}$ Help (Html5)⇔Help (Java)

a. Sketch $f(x)$.

b. For what values of x is $f(x)$ NOT continuous?