

2.3.3 What does it look like really close?

Local Linearity



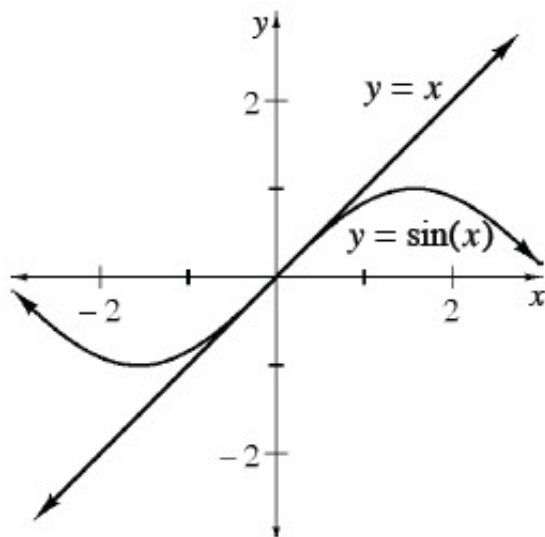
2-122. Investigate the local behavior of $y = \frac{1}{x}$ at $x = 0.5$ by graphing the function on your calculator and zooming in.

- What does the graph look like? Sketch the graph before and after you zoom in.
- Since the graph of $f(x) = \frac{1}{x}$ resembles a line in a small "local" region, we say the function is **locally linear**. Is $f(x) = \frac{1}{x}$ truly linear close to $(0.5, 2)$? Why or why not?

2-123. Study the list of basic functions below. Although the graphs vary widely by shape, some look exactly the same when you zoom in very close. What does each function look like when you zoom in at $x = -2$, $x = 0$, and $x = 4$? Record your observations.

- $y = x^2$
- $y = \sin x$
- $y = 1 - \cos x$
- $y = x^3$
- $y = |x|$
- $y = \frac{x}{x+1}$
- You probably noticed that $y = |x|$ does not have a local linearization at $x = 0$ because at $x = 0$ it has a **cusp**. What do you think the term "cusp" means and why do you think functions cannot be linearized at a cusp?

2-124. Examine the linearization of $y = \sin x$ at $(0, 0)$. Compare the values of $y = \sin x$ and the line $y = x$ at



values close to $x = 0$.

- a. Complete the table below for various x -values.

x	-1	-0.1	-0.01	0	0.01	0.1	1
$y = \sin x$							
$y = x$							

- b. For what values of x is the line a good approximation of $y = \sin x$? Write a statement summarizing your findings.
- c. On what domain is $y = x$ an over-approximation? On what domain is it an under-approximation? You should be able to *see* this information on both the graph and the table.
- d. Estimate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.
- e. Now estimate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$.

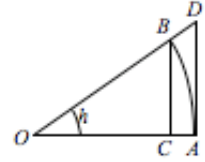
MATH NOTES



Essential Limits of Trigonometric Functions

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = 0$$

In the diagram at right, \widehat{AB} is the arc of a circle of radius 1 centered at "O", $\angle AOB$ is a small positive angle of measure h (radians) with BC and AD perpendicular to OA .



Proof 1:

Since $BC = \sin(h)$, $OC = \cos(h)$, and $AD = \tan(h)$, and since

$\text{area}(\triangle OCB) < \text{area}(\text{sector } OAB) < \text{area}(\triangle OAD)$, we can write

$\frac{1}{2} \sin(h) \cos(h) < \frac{1}{2} h < \frac{1}{2} \tan(h)$. Dividing $\frac{1}{2} \sin(h)$, we have $\cos h < \frac{h}{\sin(h)} < \frac{1}{\cos(h)}$. Taking reciprocals, $\frac{1}{\cos(h)} > \frac{h}{\sin(h)} > \cos h$. Since $\lim_{h \rightarrow 0^+} \cos h = \lim_{h \rightarrow 0^+} \frac{1}{\cos(h)} = 1$ and since $\frac{\sin(h)}{h}$ is between the two values for all $h > 0$, the limit of $\frac{\sin(h)}{h}$ is "squeezed" between the limits and must equal 1. Since $\frac{\sin(h)}{h}$ is an even function, we also see that $\lim_{h \rightarrow 0^-} \frac{\sin(h)}{h} = 1$. Thus, $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$.

Proof 2:

$$\frac{1 - \cos(h)}{h} = \frac{1 - \cos^2(h)}{h(1 + \cos(h))} = \frac{\sin^2(h)}{h(1 + \cos(h))} = \frac{\sin(h)}{h} \cdot \frac{\sin(h)}{1 + \cos(h)}$$

$$\text{Hence, } \lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{1 + \cos(h)}$$

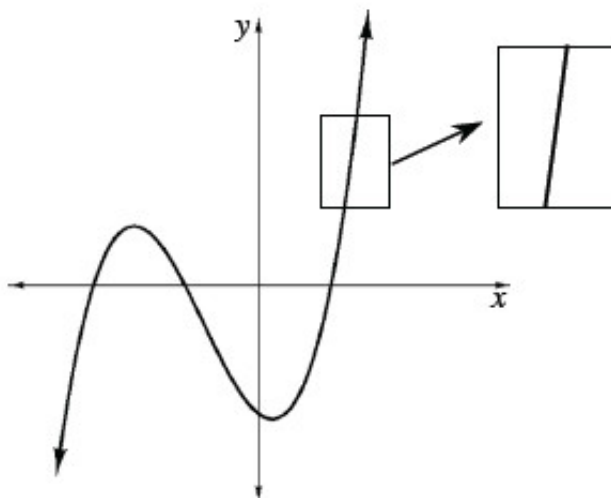
By (1), the first limit is 1, and the second limit is $\frac{0}{2} = 0$. Hence $\lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = 0$.

MATH NOTES



Local Linearity

While zooming in on a particular point of a function, often the curvature of the function becomes less noticeable as seen in the example below. However, that the curve is never exactly linear, no matter the zoom factor (unless, of course, the function was a line to begin with!)



When the graph of a function, $f(x)$, appears linear near a point, p , that line can be used to approximate or model the function's behavior near that point. This line is called the **linearization** of $f(x)$ at p .



2-125. Determine if the following functions are even, odd, or neither. Explain how you determined your choice.

[Help \(Html5\)](#) \Leftrightarrow [Help \(Java\)](#)

a. $y = \sin^2 x$

b. $y = \frac{x^2 + 1}{x^3 - 2x}$

2-126. The rate of customers who pass through the checkout stand at a grocery store depends on the time of day. Assume the rate follows the following piecewise function, where x represents the time of day (in hours) after 9:00 a.m. and $f(x)$ is the number of customers served per hour. [Help \(Html5\)](#) \Leftrightarrow [Help \(Java\)](#)

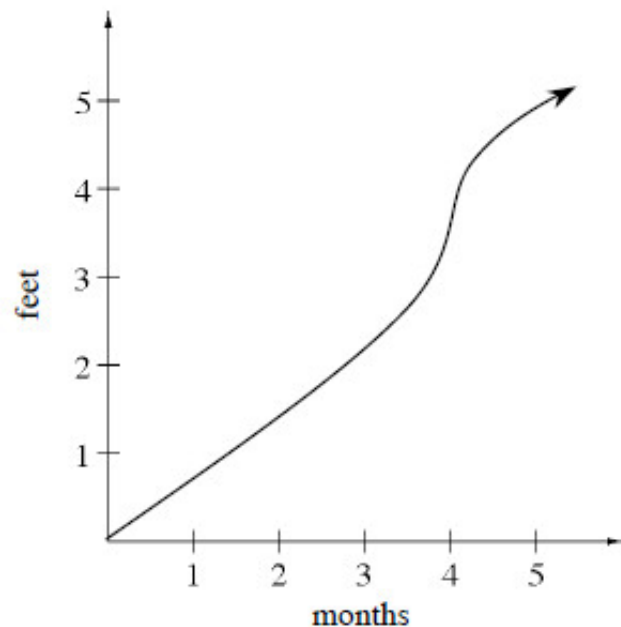
$$f(x) = \begin{cases} 120x & \text{for } 0 \leq x < 2 \\ 180x - 120 & \text{for } 2 \leq x < 5 \\ 240x - 420 & \text{for } x \geq 5 \end{cases}$$



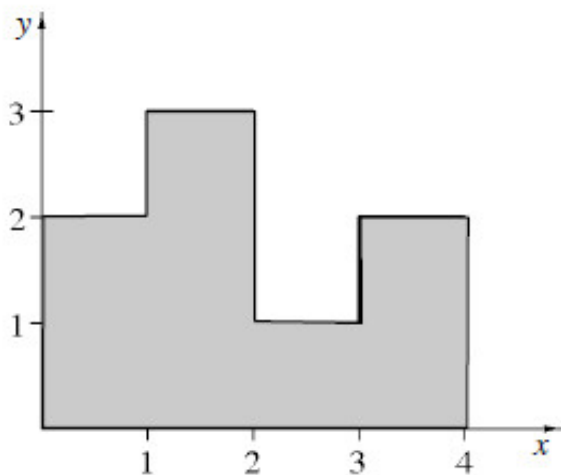
- Graph the function and name its domain and range.
- Is this function continuous?
- Find $A(f, 0 \leq x \leq 8)$. What are the units of this area? What does this area represent?

2-127. Fertilizer needs to be applied during the fastest growth of the plant. Below right is the graph of the growth cycle of a flowering shrub. [Help \(Html5\)](#) \Leftrightarrow [Help \(Java\)](#)

- Using complete sentences, write a detailed statement describing the growth of this shrub for $0 \leq t \leq 5$ months.
- During which time t should the plant be fertilized?
- Approximately how fast is the shrub growing at $t = 3$? How did you get your answer?
- What is the shrub's *average* rate of growth over the complete growth cycle? How did you get your answer?



2-128. Rotate the flag shown below (created by the region under the curve for $0 \leq x \leq 4$) about the x -axis. Describe (and draw) the shape that is created and find its volume. [Help \(Html5\)](#) \Leftrightarrow [Help \(Java\)](#)



2-129. What is the relationship between the slopes of perpendicular lines? If you know the slope of one of the lines, how can you find the slope of a line perpendicular to it? [Help \(Html5\)](#) \Leftrightarrow [Help \(Java\)](#)

2-130. A function, f , is *continuous* for all real numbers. If $f(x) = \frac{x^2-9}{x+3}$ when $x \neq -3$, then what must $f(-3)$ equal? Write a piecewise function that represents this situation. [Help \(Html5\)](#) \Leftrightarrow [Help \(Java\)](#)

2-131. Evaluate the following limits: [Help \(Html5\)](#) \Leftrightarrow [Help \(Java\)](#)

a. $\lim_{x \rightarrow \infty} \frac{\sqrt{9x^2 - 3x + 1}}{4x^2 - 1}$

b. $\lim_{x \rightarrow -1} \frac{x+1}{x^2+5x+6}$

c. $\lim_{x \rightarrow -\infty} \frac{2-3x-4x^2}{(1-3x)^2}$

d. $\lim_{x \rightarrow 2} \frac{|x^2-4|}{x-2}$