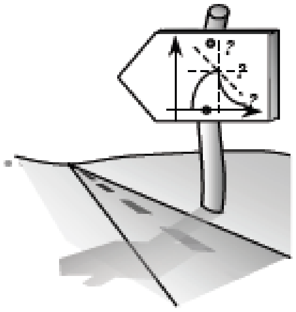
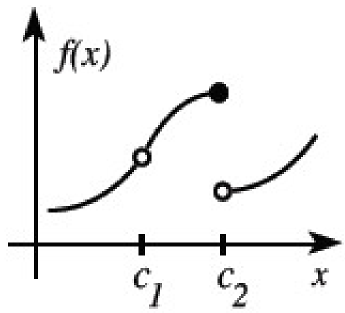
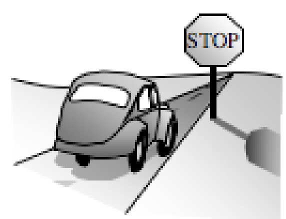


3.4.2 What's up with a cusp?

Curve Constructor: Part II

3-156. Explain why a function must be continuous at $x = c$ to be differentiable at $x = c$. The graph below may help you.





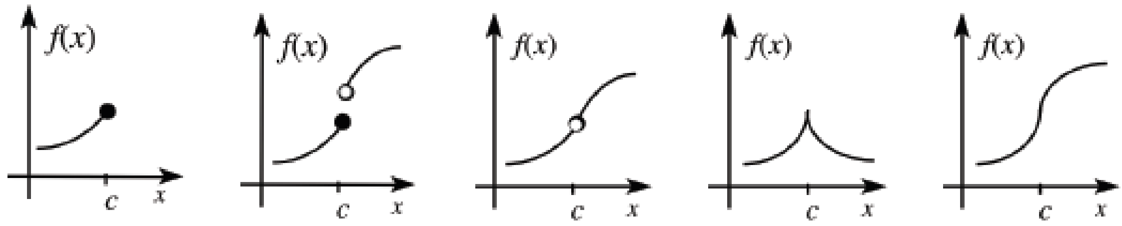
MATH NOTES

Intuitive Definiton of Differentaibility

If the curve is continuous and has a local linearization at a point (c, f(c)) then the function is **differentiable** at (c,$f(c)$). This means that a non-vertical line can be drawn tangent to

$f(x)$ at $x = c$.

The following functions are not differentiable at $x = c$. They each fail at least one of the conditions of differentiability listed above. Notice that the function in Graph A below has an endpoint at $x = c$. This means that the limit of $f(x)$ as $x \to c^+$ does not exist, so the limit as $x \to$c cannot exist either.



Graph A **Graph B** **Graph C** **Graph D** **Graph E**

So far, we have studied several functions that have a "sharp point" in its graph, such as Graph D above. We call this shape a **cusp.** It occurs when the left-hand and right-hand derivative limits disagree.

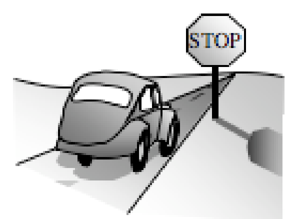
3-157. FUNKY FUNCTIONS, Part One

a. Graph $f(x) = 2 + (0.1 - |x|)^2$ and rewrite $f(x)$ as a piecewise function.

b. Zoom-in at $x = 0$ on your graphing calculator and carefully examine the left- and right-hand derivatives. Does $f(x)$ appear differentiable at $x = 0$? Why or why not?

c. To confirm whether or not $f(x) = 2 + (0.1 - |x|)^2$ is differentiable at $x = 0$, we need to examine $f'(x)$. Use the piecewise function from part (a) to find $f'(x)$ for $x \neq 0$.
Analyze $\lim\limits_{h \to 0^-} \frac{f(x+h)-f(x)}{h}$ and $\lim\limits_{h \to 0^+} \frac{f(x+h)-f(x)}{h}$. Do they agree?

Math Notes



Formal Definition of a Derivative

A function $f(x)$ has a **derivative** (is **differentiable)** at the point $x = p$ if $\lim\limits_{h \to 0} \frac{f(p+h)-f(p)}{h}$ exists as a finite number and L is the value of this limit. We write $f'(p) = L$.

We say $f(x)$ is differentiable on an interval if it has a derivative $f'(x)$ at every point of the interval.

Note that $f'(x)$ exists whenever $f(x)$ is smooth at x. Typical points where $f'(x)$ does not exist are values of x where the graph of $f(x)$ is not continuous or has a cusp. For example, if $f(x) = |x|$, then $f'(0)$ does not exist.

3-158. Use the *definition of the derivative as a limit* to find the slope function, $f'(x)$ of $f(x) = 4x^2 - 3$. Then use your slope function to find $f'(11)$ and $f'(1000)$.

3-159. ABSOLUTE VALUE

a. Graph $y = |x|$ on graph paper and *without your calculator,* sketch $\frac{dy}{dx}$.

b. What happens to $\frac{dy}{dx}$ at the vertex? Verify your observations by examining the slopes on both sides of the vertex.

c. Use your graphing calculator to find the slope of $y = |x|$ at the vertex. What happened?

d. Part of the reason most graphing calculators incorrectly determine slopes at the vertex of an absolute value graph, as well as other cusps, is because they use the **symmetric difference quotient** (Hanah's Method) to calculate the slope of a tangent.
For $f(x) = |x|$, use $\frac{f(x+h)-f(x-h)}{2h}$ to calculate $f'(0)$ for $h = 0.1, -0.1$, and 0.01. What do you notice? What leads the calculator to falsely find a derivative of $f(x) = |x|$ at $x = 0$?

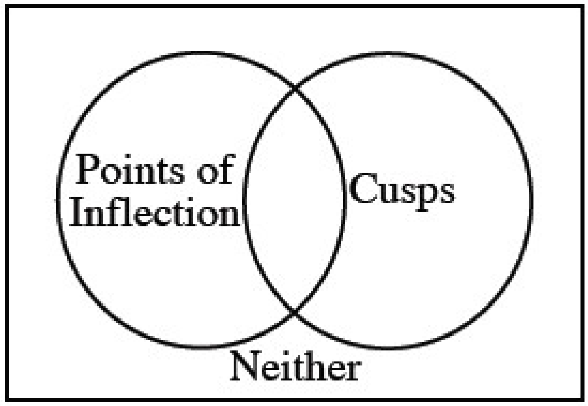
3-160. CURVE CONSTRUCTOR, Part Two
Revisit our computer graphics program from 3-82. By using our arc tool, we can make four different types of arcs, shown below.

a. The software user can use the arc tool twice and then connect the curves to make a continuous curve, such as that shown below. Draw every possible combination of two of these arcs together.

b. Which of these combinations must create a curve with a cusp?

c. Which of these combinations has a point of inflection?

d. Place the sixteen combinations appropriately in the Venn diagram below.



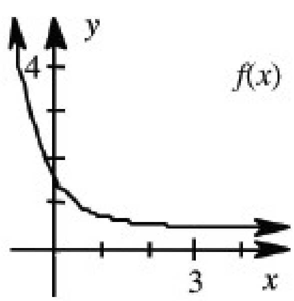
3-161. Find the equations of the lines tangent to the curve $y = x^3 - 4x$ at both $x = 0$ and $x = 2$. Then, determine the point of intersection for these two tangent lines. Help (Html5)⇔Help (Java)

3-162. For each graph below: Help (Html5)⇔Help (Java)

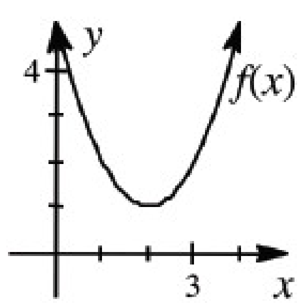
i. Trace $f(x)$ on your paper and write a slope statement for $f(x)$.

ii. Sketch the graph of $f'(x)$ using a different color.

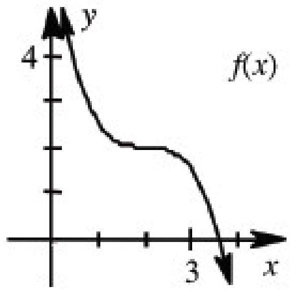
a.



b.



c.



3-163. Write and evaluate a Riemann sum to estimate $A(y,\ 1 \le x \le 8)$ for $y = \frac{4}{x}$ with 10 left endpoint rectangles. Help (Html5)⇔Help (Java)

3-164. Draw the graph of $f(x)$ given the following information about its slope function, $f(x)$. Help (Html5)⇔Help (Java)

- $f'(x) > 0$ on $1 < x < 4$
- $f'(x) < 0$ on $x < 1$ and $x > 4$
- $f'(x) = 0$ for $x = 1$, $x = 4$

3-165. Find $\frac{dy}{dx}$ for each of the following equations. You will need to rewrite each equation first. Help (Html5)⇔Help (Java)

a. $y = \sqrt[3]{\frac{1}{x^2}}$

b. $y = x\sqrt{x}$

c. $y = \sin^2 x + \cos^2 x$

d. $y = \frac{x+2}{x}$

3-166. Find the general antiderivative $F(x)$ for each function below. Test your solution by verifying that $F'(x) = f(x)$. Help (Html5)⇔Help (Java)

a. $f(x) = 3x^{1/2} - 7x$

b. $f(x) = \cos x + 2 \sin x$

3-167. Define $f(x)$ and $g(x)$ so that $h(x) = f(g(x))$, for the following functions given that $f(x) \neq x$ and $g(x) \neq x$. Help (Html5)⇔Help (Java)

a. $h(x) = \cos(3x - 11)$

b. $h(x) = \sqrt[3]{2^x - 1}$

c. $h(x) = 3^{5-2x}$

3-168. Evaluate each limit. If the limit does not exist, say so but also state if y is approaching positive or negative infinity. Help (Html5)⇔Help (Java)

a. $\lim_{x \to 0} \frac{\sqrt{3-x}-\sqrt{3}}{x}$

b. $\lim_{x \to 2} \frac{x+3}{x-4}$

c. $\lim_{x \to 2^+} \frac{x^2|x-2|}{x-2}$

d. $\lim_{x \to \infty} e^{-x} + 1$

3-169. Use the *definition of a derivative* as a limit to find $f'(x)$ if $f(x) = 2x + 9$. Use the Power Rule to confirm your solution. Help (Html5)⇔Help (Java)