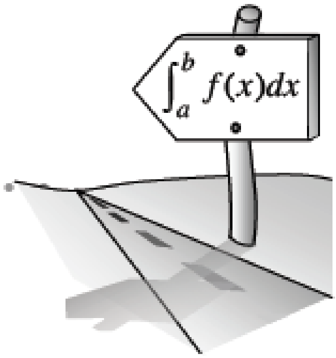
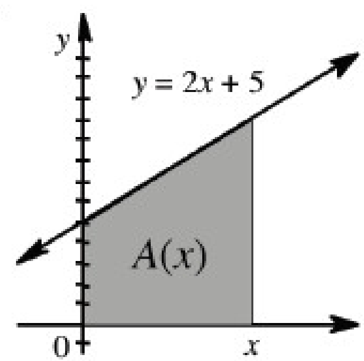
4.2.3 What is the FTC?



The Fundamental Theorem of Calculus

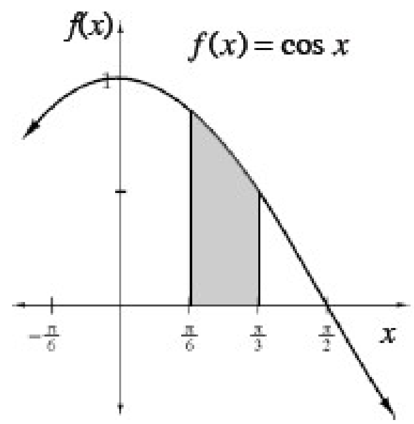


4-65. Anita was doing her homework and noticed area functions and antiderivatives were very similar. Find $A(x)$ and $F(x)$ for some simple functions. What do you notice?

i. $f(x) = 2$

ii. $f(x) = 2x$

4-66. Tommy and Anita have a problem. They want to find the area under $f(x) = \cos x$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ but they do not have a calculator.



a. Set up an integral to represent the situation.

b. Tommy knows that an integral is an area and the integral could be evaluated by $A\left(\frac{\pi}{3}\right) - A\left(\frac{\pi}{6}\right)$. Unfortunately, he does not know the area function for $f(x) = \cos x$. Give Tommy advice about what to do.

c. Anita remembers her homework from last night and suggests using $F(x)$, the antiderivative, instead of $A(x)$ to solve this problem. What is $F(x)$ for $f(x) = \cos x$?

d. Anita proceeds with her plan. She sets up the problem as follows:

$$\int_{\pi/6}^{\pi/3} \cos x dx = A\left(\frac{\pi}{3}\right) - A\left(\frac{\pi}{6}\right) = F\left(\frac{\pi}{3}\right) - F\left(\frac{\pi}{6}\right)$$

Finish her work. Then use a calculator to check your answer.

e. Tommy and Anita are delighted by their fundamental discovery. Using antiderivatives, they now have a procedure to find the exact area under the curve of any function! Even though antiderivatives have a $+ C$ while area functions do not, the C is eliminated when evaluating definite integrals. Explain why.

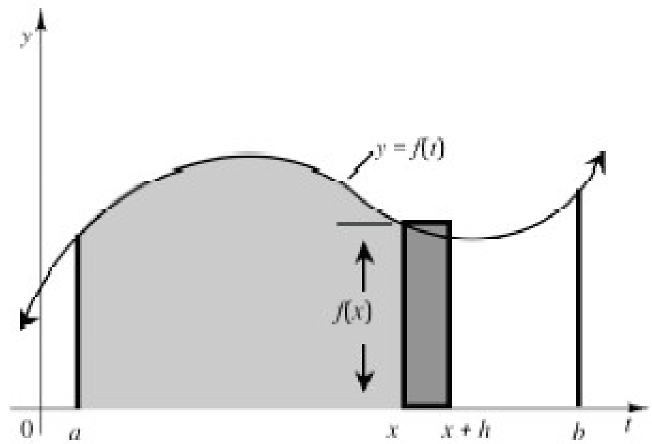
4-67. THE FUNDAMENTAL THEOREM OF CALCULUS, Part One

a. Use geometry to find the general area function $A(x)$ for $f(x) = 2x + 5$. That is, find $\int (2x+5)dx$.

b. Find $A'(x)$ and compare it to the original function.

c. Part (b) shows that the derivative of an area function is the original function. Test this idea on a general linear equation, $f(x) = mx + b$, by finding $A(x)$ and $A'(x)$. Does the same result happen?

4-68. Now we will explore the general relationship between a function f, its area function A, and its antiderivative F.



a. Using the picture at right, explain why as x changes, $A(x)$ changes. Use geometry to explain why $A'(x) = f(x)$.

b. Now examine $A'(x)$ analytically.

If $A(x) = \int_c^x f(t)dt$ then:

$$A'(x) = \frac{d}{dx}\int_c^x f(t)dt$$

$$A'(x) = \frac{d}{dx}(F(x) - F(c))$$

$$A'(x) = \frac{d}{dx}(F(x)) - \frac{d}{dx}(F(c))$$

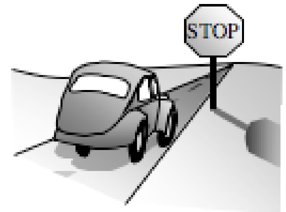
How can we further simplify $\frac{d}{dx}(F(x)) - \frac{d}{dx}(F(c))$?

c. How does the diagram above demonstrate your results from part (b)?

d. You have just demonstrated that the derivative of the integral of f is the original function of f. How about the reverse - what is the integral of the derivative of f (such as $\int f'(x)dx$)?

4-69. Your conclusion from part (b) of problem 4-68 is an example of the most important theorem you will learn all year: the Fundamental Theorem of Calculus (FTC) stated carefully in the Math Notes box below. According to the Math Note, it seems that for any function, $f(x)$, the general area function, $A(x)$, and the antiderivative, $F(x)$, are related. Describe this relationship.

MATH NOTES



The Fundamental Theorem of Calculus

Part 1 If $f(x)$ is continuous on $[a, b]$ and $F(x)=\int_a^x f(t)\,dt$, then $F'(x)=f(x)$ for all x in (a, b).

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Part 2 If $F(x)$ is any function such that the derivative $F'(x)$ exists for all x in an open interval containing $[a, b]$ and $F'(x)=f(x)$ for all x in $[a, b]$, then $\int_a^b f(x)\,dx = F(b)-F(a)$.

4-70. Use the Fundamental Theorem of Calculus to evaluate $\int_1^2 (6x^2+7)\,dx$. Test your results using a graphing calculator.

4-71. Find the following.

a. Find $\int_0^x (2x+5)\,dx$.

b. Find $\int_4^9 (2x+5)\,dx$.

4-72. When evaluating an integral like $\int_4^9 (2x+5)\,dx$, we often use the following notation:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a)$$

Using the notation above, evaluate $\int_4^9 (2x+5)\,dx$.

Review & Preview

4-73. Rewrite the following integral expressions using a single integral. Help (Html5)⇔Help (Java)

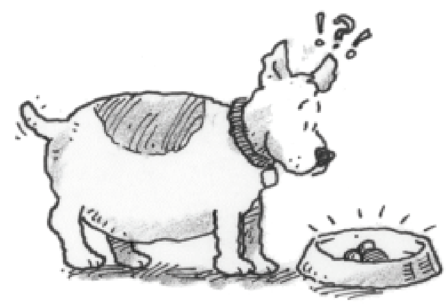
a. $\int_{-5}^{2} 6\,x^3 dx - \int_{-5}^{2} -9x\,dx$

b. $\int_{5}^{9} h(x)\,dx + \int_{9}^{3} h(x)\,dx$

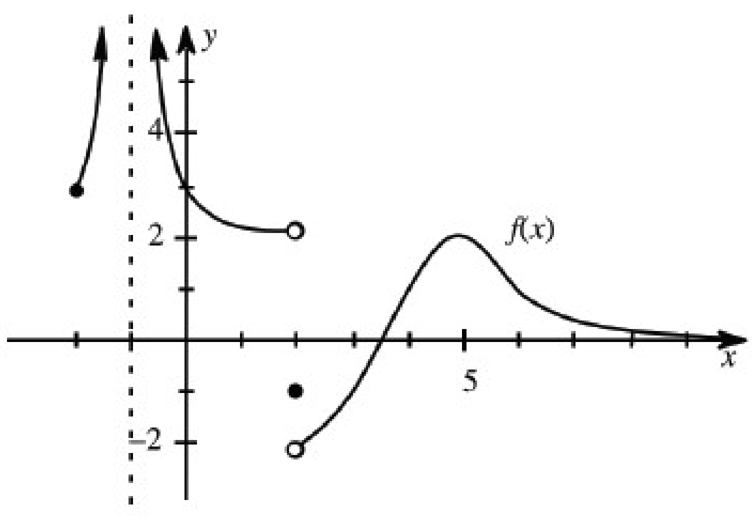
c. $\int_{2}^{7} \pi\,(2x)^2\,dx + \int_{2}^{7} \pi(\sqrt{x})^2\,dx$

d. $\int_{1}^{4} (x+5)^2\,dx + \int_{6}^{9} \sqrt{x}\,dx$

4-74. Keily put his dog on a diet! If $f(x)$ is measured in pounds lost per day and t is measured in days, what does $\int_{a}^{b} f(t)\,dt$ represent? What are its units? Help (Html5)⇔Help (Java)



4-75. The graph of a function $f(x)$ is given at right. Use the graph to evaluate the following limits. Help (Html5)⇔Help (Java)



a. $\lim\limits_{x \to -1} f(x)$

b. $\lim_{x \to 2} f(x)$

c. $\lim_{x \to 2^-} f(x)$

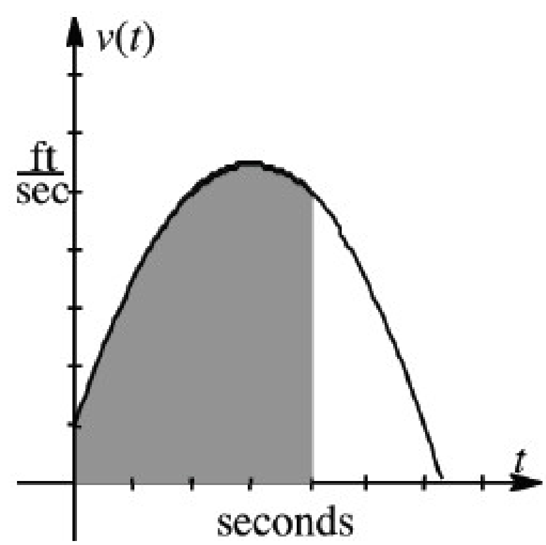
d. $\lim_{x \to 2^+} f(x)$

e. $\lim_{x \to 5} f(x)$

f. $\lim_{x \to \infty} f(x)$

g. Where (if anywhere) does the derivative of $f(x)$ not exist?

4-76. The graph at right shows the velocity of an object over time defined by the function $v(t) = -0.5t^2 + 3t + 1$. Help (Html5)⇔Help (Java)



a. Use your graphing calculator to evaluate

$$\int_0^4 (-0.5t^2 + 3t + 1)\,dt$$

b. What does the result in part (a) find?

c. What does $\frac{d}{dt}(v(t))$ represent?

d. What would the units be in part (c)?

4-77. If $\int_2^4 f(x)dx = 10$ find: Help (Html5)⇔Help (Java)

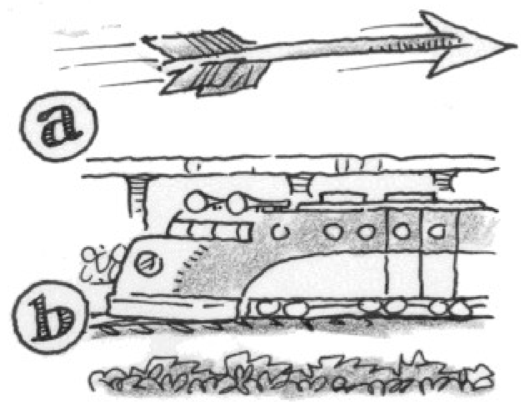
a. $\int_4^2 -f(x)dx$

b. $\int_{10}^{12} (f(x-8)+4)dx$

c. $\int_{10}^{12} f(x+8)dx$

4-78. Given $f(x) = 2x$, find the equation of a vertical line that would divide $\int_0^{10} f(x)\,dx$ in half. Help (Html5)⇔Help (Java)

4-79. Compare how distance and velocity are related with these two scenarios: Help (Html5)⇔Help (Java)

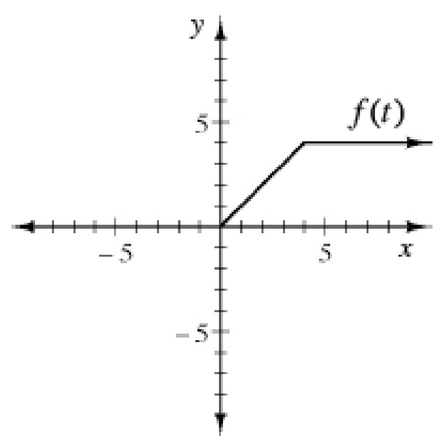


a. As an arrow flies through the air, the distance it has traveled in feet at time t is $s(t) = 4\sqrt{t}$. *Without your calculator*, find the velocity, $s'(t)$, at times $t = 1$, 4, and 16 seconds. Explain what concepts of calculus you applied in order to solve this problem.

b. As a train travels through a station, its velocity, measured in miles per hour, is $v(t) = 9t + 32$. If the train is in the station when $t = 0$, determine the position of the train at $t = 1$ hour. Explain what concepts of calculus you applied in order to solve this problem.

c. Both (a) and (b) involve distance and velocity. However, each required a different method or approach. Describe the relationship between distance and velocity, as well as the derivative and area under a curve.

4-80. Use the partial graph of f(t) shown at right to complete the following questions. Help (Html5)⇔Help (Java)



a. If $f(t)$ is an odd function and $g(x) = \int_0^x f(t)\,dt$, evaluate:

i. $g(-3)$

ii. $2 \cdot g(-7)$

iii. $g(9) + g(-9)$

b. If $f(t)$ is an even function and $g(x) = \int_0^x f(t)\,dt$, evaluate:

i. $g(-3)$

ii. $2 \cdot g(-7)$

iii. $g(9) + g(-9)$