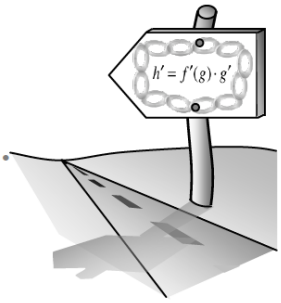
5.2.1 Can you prove the Product Rule?



The Product Rule

5-47. Today you are going to find the derivative of functions of the form $j(x) = f(x) \cdot g(x)$. Some of these are easy to do using algebra.

a. Find the derivative of:

   i. $j(x) = x(x + 3)$

   ii. $j(x) = (x + 10)(x + 5)$

   iii. $j(x) = (x - 1)^2$

b. Do you notice any patterns?

c. Try some more difficult cases:

   i. $j(x) = (2x + 1)(x + 5)$

   ii. $j(x) = (5x - 1)(x + 10)$

   iii. $j(x) = (3x^2 + 1)(-x^3 + 2)$

d. What do you notice?

e. Let's consider a function that is not easy to expand: $j(x) = x \sin x$. It would be nice to develop a method to differentiate the products of two functions. Explain how the functions above can be modeled as $j(x) = f(x) \cdot g(x)$.

5-48. When Hana looked at the function $j(x) = f(x) \cdot g(x)$ she immediately thought of a rectangle whose length and width were $f(x)$ and $g(x)$ and area was $j(x)$. Hana would like to find $j'(x)$ by using the definition of the derivative as a limit.

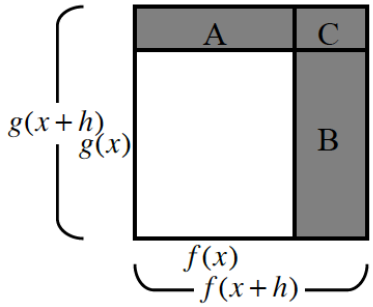
a. If $j(x)$ represents the area of rectangle, what would $j'(x)$ represent.

b. To determine the value of $j'(x)$, she set up the problem as follows:

$$j'(x) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

Hana sketches a rectangle in which the shaded area represents the numerator. Examine her diagram and explain why:

$f(x+h)g(x+h) - f(x)g(x) = (\text{area A}) + (\text{area B}) + (\text{area C})$



c. Recall that Hana wants to know the derivative of $j(x) = f(x)g(x)$. So she rewrites the definition as $j'(x) = \lim_{h\to 0} \frac{(\text{area A})+(\text{area B})+(\text{area C})}{h}$. What did she do?

d. She decides to evaluate each limit separately and add the results. Justify each of her steps for area A.

$$\lim_{h\to 0} \frac{\text{area A}}{h} = \lim_{h\to 0} \frac{f(x)[g(x+h)-g(x)]}{h} = \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

Now simplify.

e. Justify the steps for area B.

$$\lim_{h\to 0} \frac{\text{area B}}{h} = \lim_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h} = \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Now simplify.

f. Justify the steps for area C.

$$\lim_{h\to 0} \frac{\text{area C}}{h} = \lim_{h\to 0} \frac{[f(x+h)-f(x)][g(x+h)-g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} [g(x+h)-g(x)] = f'(x)\cdot 0 = 0$$

g. Use substitution to express $j'(x)$ as a function of x.

h. Use your result to differentiate $j(x) = (4x - 1)(3x + 1)$, verify the derivative using algebra and the Power rule. Are the results the same?

5-49. The derivative technique you summarized in problem 5-48 part (e) is called the **Product Rule**. It seems to work for simple binomials like $f(x) = (4x - 1)(3x + 1)$, but what about other kinds of functions?

a. Differentiate $y = x \sin x$.

b. In this case, you cannot check your derivative the same way you did for the previous problems. Find another way to verify your result.

c. Repeat this process for $y = (x - 2)^2 (x + 5)^3$.

5-50. SNEAKY PRODUCT RULE PROBLEMS

Sometimes a complicated equation can be differentiated by first rewriting it as a product. Since the Product Rule is often easier than other techniques, recognizing this strategy will come in handy later.

Find the derivative of each equation below by first rewriting each as a product of two differentiable functions.

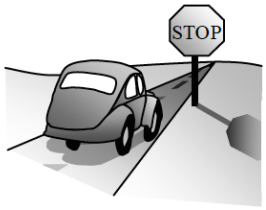
a. $y = (5x^2 + 8)^2$

b. $y = \frac{5}{(x-8)^2}$

c. $y = \frac{x^3}{(x-8)^2}$

d. $f(x) = \sin^2 x$

MATH NOTES



The Product Rule

If $f'(x)$ and $g'(x)$ exist and $j(x) = f(x) \cdot g(x)$, then $j'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

Refer to problem 5-48 for a proof.

5-51. Evaluate the following integrals *without a calculator.* Homework Help ✎

a. $\int_1^4 \left(3\sqrt{u} + \frac{1}{\sqrt{u}}\right) du$

b. $\int_0^2 6(x-4)^2 \, dx$

c. $\int \frac{d}{dx}[g(x)] dx$

d. $\int_0^1 k^{5/3}\,dk$

e. $\int_1^2 \frac{4}{n^3}\,dn$

5-52. Differentiate if possible. If the function is one we have not yet studied, say so and explain how you know. Be sure to use the appropriate notation. Homework Help ✎

a. $y = x^{100} + 100x$

b. $H(t) = t^{-4} - 6\sqrt{t}$

c. $y = x \cdot 2^x$

d. $y = x^8 \cos x$

e. $f(x) = x^5\,(x^3 + \pi)$

f. $y = \frac{7}{x} - \frac{x}{7}$

5-53. Let $f(x) = x^3$, $g(x) = 6x - 1$, and $h(x) = \sin x$. Match each function in the left column of the table with its expression in the right column. Homework Help ✎

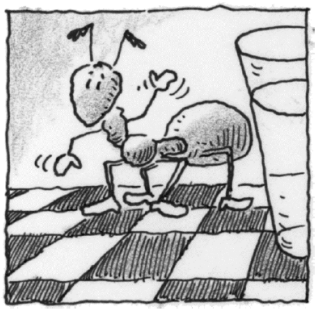
a.	$f'(x)$	A.	$3(6x - 1)^2$
b.	$g'(x)$	B.	6
c.	$h'(x)$	C.	$3\sin^2 x$
d.	$f'(g(x))$	D.	$\cos(x^3)$
e.	$g'(f(x))$	E.	$3x^2$
f.	$f'(h(x))$	F.	1
g.	$h'(f(x))$	G.	$\cos x$

5-54. There is an ant moving along the edge of your table. Its position from the center of one edge of the table is given by the equation $x(t) = (t - 1)(t - 4)^4$. When $x(t)$ is increasing, the ant is moving forward. Homework Help ✎

a. Find $\frac{dx}{dt}$.

b. If $x(t)$ represents how many millimeters the ant is from the center of the table's edge after t seconds, what are the units for $\frac{dx}{dt}$? What information does this derivative give about the ant?

c. What are the units for $\frac{d^2x}{dt^2}$? What information does this second derivative give about the ant?



d. How often does the ant stop and change direction? At what time(s)? Be sure to find and confirm your answer using both calculus techniques and a graph.

e. At time $t = 3$, is the ant moving forwards or backwards? Justify your answer using calculus.

5-55. Draw the graph of $f(x) = \sqrt{x}\cos x$ for $0 \le x \le 3$. Homework Help ✎

a. Find $f'(x)$.

b. Graph $f'(x)$ using your graphing calculator. Then, use this graph to find the x-value of the local maximum of $f(x)$.

c. Likewise, use a graph of $f''(x)$ to find the point of inflection of $f(x) = \sqrt{x}\cos x$ between $0 \le x \le 3$.

5-56. Find all values of x for which the slope of the line tangent to $y = \frac{1}{6}x^3 - 4x^2 + 9x - 1$ is 5.

Homework Help ✎

5-57. After a weather balloon is launched the following data is sent back to the meteorological center. The column marked time, t, is given in seconds after launch, elevation, e, in feet above sea level, and temperature, T, in degrees Fahrenheit.

Homework Help ✎

t (Time)	e (Elevation)	T (Temp)
0	1260	56.4°
30	1560	55.1°
60	1920	53.9°
90	2350	51.9°
120	2750	49.5°
150	3170	47.7°
180	3600	45.0°
210	4080	42.4°
240	4560	40.9°

a. Approximately how fast (in ft/sec) is the balloon rising at 120 seconds?

b. Approximately how fast is the temperature changing (in °F/ft) at 2750 feet?

c. Approximately how fast is the temperature changing (in °F/sec) at 120 seconds?

d. Why would you expect the product of the answers to (a) and (b) to equal answer (c)?

5-58. Answer the questions (a) through (c) of the previous problem when $t = 210$ seconds and $e = 4080$ feet. Homework Help ✎