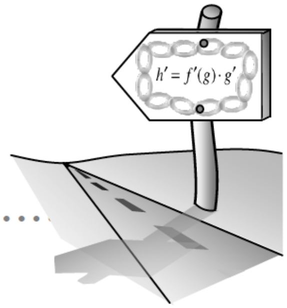
5.2.2 Can I differentiate composite functions?



Chain Rule and Application: Part I

5-59. Weather Records Melt!

Weather records that had stood for generations melted like butter Wednesday as a fierce heat wave roasted the Bay Area…. At 106°, it was so hot in Fairfield that part of the pavement on I-80 buckled in the heat, forcing Caltrans to close the three left lanes near North Texas Street.

a. Why did this happen?

b. The pavement length depends on temperature, and temperature in turn depends on time of day. Explain why this is an example of a composite function.

c. Explain how the length of the pavement changes during the day. What does the rate of change of the length depend on?

5-60. THE BUCKLED FREEWAY PROBLEM
The buckled freeway is an example of a composite function: the road length depends on temperature, and temperature in turn depends on time of day.

Let's study how the rate of change of the composite function depends on the rates of change of its component functions. These tables show temperatures and length of a section of road on a particular day. Use these tables to approximate the rates of change.

For example, to estimate how the temperature is changing at exactly 4 p.m., we could find:

$h'(4) = \frac{80-83}{5-3} = -1.5$ degrees per hour.

Approximate each of these rates:

a. The rate of temperature increase at 3 p.m.

b. The rate of length increase per degree when the temperature is 83°.

c. The rate of length increase per hour at 3 p.m.

d. The rate of length increase per hour at 5 p.m.

Temperature during day

Time t	Temp $x = h(t)$
1 pm	74º
2 pm	79º
3 pm	83º
4 pm	84 º
5 pm	80º
6 pm	75 º
7 pm	69º

Lengths at certain temperatures

Temp x	Length (ft) $y = L(x)$
68º	50.0
69º	50.1
70º	50.2
71º	50.3
72º	50.5
73º	50.7
74º	50.9
75º	51.2
76º	51.5
77º	51.8
78º	52.2
79º	52.6
80º	53.1
81º	53.6
82º	54.1
83º	54.7
84º	55.3
85º	55.9

5-61. DERIVATIVES OF COMPOSITE FUNCTIONS
You have been working with composite functions for a while: $f(x) = (x-6)^3$, $f(x)=\sqrt{5x+2}$ and $f(x) = \sin(3x)$ are three examples. All three can be written in the form $f(x) = g(h(x))$, the composition of an "inside function" and an "outside function."

Today you will develop a technique to *differentiate*composite functions.

a. As we found in Chapter 4, if $f(x) = (x + 5)^2$, then $f'(x) = 2(x + 5)$ because $f(x) = (x + 5)^2$ is a horizontal translation of $f(x) = x^2$.

What if $f(x) = \sin 3x$? Graph this function in $Y_1$of your calculator. How does the horizontal transformation affect the graph? Think about the slopes of f and make a conjecture about $f'(x)$. Graph your conjecture in Y_2.

b. Your teacher will show you how to input the graph of the actual derivative, $f'(x)$ in Y_3 of your calculator. Does the actual graph match your conjecture? If not, adjust your conjecture until it matches.

c. Explain why the derivative you found in part (b) makes sense graphically.

5-62. Let's try this with a more complicated composite function: $g(x) = \sin x^2$.

a. Compare the graphs of $y = \sin x$ to $y = \sin x^2$. What does the inner function (x^2) do to the shape of the graph?

b. Let $g(x) = \sin x^2$, make a conjecture about the derivative of this function.

c. Graph $g'(x)$ and your conjecture in your calculator. Adjust your conjecture until the two graphs match.

d. Compare $\frac{d}{dx}(\sin x)$ with $\frac{d}{dx}(\sin x^2)$. Explain what the inner function (x^2) does to the shape of the derivative graph.

5-63. Based on what you have just learned, find $g'(x)$ if $g(x) = \sin 3x^2$.

5-64. Let $y = \sin u$ and let $u = 5x^2$. We want to find $\frac{dy}{dx}$.

a. Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

b. Explain why $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, then use your answers from part (a) to find $\frac{dy}{dx}$.

c. What you have just discovered is called the "Chain Rule." Why do you think we give it this name?

Math Notes

The Chain Rule

The Chain Rule allows us to differentiate composite functions.

If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$.

A careful proof of this result is beyond the scope of this course.

5-65. Find $\frac{d}{dx}((3x + 5)^2)$ in two ways: first by using the Chain Rule and then by using the Product Rule. Verify that your answers match.

5-66. Explain why $f'(x) = 2(3x + 5) = 6x + 10$ is incorrect given $f(x) = (3x + 5)^2$. What does it equal?

5-67. Differentiate the following functions using the Chain Rule and test your result with your graphing calculator. You do not need to simplify your answer.

a. $f(x) = \sin(2x + 3)$

b. $f(x) = \sqrt{16 - x^2}$

c. $f(x) = \cos(\sin x))$

d. $f(x) = (3x^2 + 2x + 4)^{-2}$

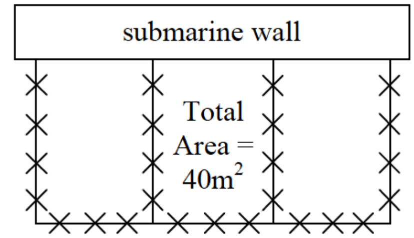
Review & Preview

5-68. Describe how to find the derivative for $f(x) = g(ax + b)$, where a and b are constants. Check by using your derivative on $f(x) = \sqrt{5x+2}$ and $f(x) = \cos(3x)$. Homework Help ✎

5-69. YELLOW SUBMARINE, Part Two

Ring's detention chamber for blue meanies worked fine at first, but a problem developed because he was too successful at catching them! Their meanness led to lots of fights, noise, and general unpleasantness,

so he needed to modify his detention chamber. In his second design, he split the chamber into three parts, with barbed wire partitions of course! The entire chamber should still have 40 square meters of area. What dimensions of the chamber minimize the amount of barbed wire?
Homework Help ✎



5-70. Today's mystery function has these properties: $f''(-3) = f'(-3) = f(-3) = 0$. What do you know about the graph? What don't you know? Homework Help ✎

5-71. Write an expression for the distance between each set of points. Homework Help ✎

a. (5, 10) and (8, 14)

b. (a, b) and (3, 5)

c. (85, −10) and (−8, 142)

d. A random point on $y = x^2$ and (3, 5), (in terms of x.)

5-72. Evaluate the following integrals *without a calculator*. Homework Help ✎

a. $\int_{-1}^{-1}(-9t^{6/5} - 15t^{-11/3})\,dt$

b. $\int_0^2 \frac{d}{dx}(9x-1)\,dx$

c. $\int(-32t+12)\,dt$

d. $\int \cos(5x)\,dx$

e. $\int_1^{-1}(4x^2-6x)\,dx + \int_1^2(2x^2-3x)\,dx$

5-73. Find the area of the region formed by $y = \sqrt{x+1}$ and $y = x^2 - 2x - 3$. Homework Help ✎

5-74. Chris found the derivative of $(2x + 5)^2$ as shown below. Homework Help ✎

$$\frac{d}{dx}((2x+5)^2) = \frac{d}{dx}(4x^2+20x+25) = 8x+20$$

To check his solution, he tried applying the Power Rule:

$$\frac{d}{dx}((2x+5)^2) = 2(2x+5) = 4x+10$$

His solutions did not match! Which is the correct derivative, and why?

5-75. Use your derivative tools to find $\frac{dh}{dx}$. Homework Help ✎

a. $h(x) = \cos(x^2)$

b. $h(x) = \sqrt[3]{\sin x}$

c. $h(x) = 6x^{2/3}$

d. $h(x) = \frac{1}{2}x^5\sqrt{x-1}$

e. $h(x) = 2\sin\left(\frac{1}{x}\right)$

f. $h(x) = -4\cos x \sin x$