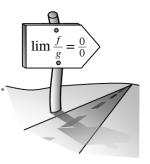# 5.5.2 Why does l'Hôpital's Rule work?

Using l'Hôpital's Rule

Yesterday you saw that the limit of an indeterminate form exists as a ratio, by comparing the way the terms change can lead to a finite value. The Math Note below summarizes this.

MATH NOTES

STOP

Indeterminate Limits And l'Hôpital's Rule

If we have an **indeterminate limit** of the form $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ (or $\frac{\infty}{\infty}$) there is a powerful tool called l'Hôpital's Rule which states:

If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

The basic idea is to look at the linearization of $f(x)$ and $g(x)$ near a. Since $f(a) = g(a) = 0$, the linearization of $f(x)$ is $0 + m(x - a)$ where $m = f'(a)$ and the linearization of $g(x)$ is $0 + n(x - a)$ where $n = g'(a)$.

Hence $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{m(x-a)}{n(x-a)} = \frac{m}{n}$ or $\frac{f'(a)}{g'(a)}$.

5-161. Try l'Hôpital's Rule on these limits, which you originally saw in Chapter 1. Remember to use l'Hôpital's only when the limit is indeterminate! If the limit is not indeterminate, use another method.

a. $\lim\limits_{x \to 0} \frac{\sin x}{x}$

b. $\lim\limits_{x \to 0} \frac{\sin^2 x}{x}$

c. $\lim\limits_{x \to 0} \frac{\sin x}{x^2}$

d. $\lim\limits_{x \to 0} \frac{\cos x}{x}$

e. $\lim\limits_{x \to 0} \frac{1-\cos x}{x-1}$

f. $\lim\limits_{x \to 0} \frac{1-\cos x}{x}$

5-162. Repeated Use of l'Hôpital's Rule

a. Aram tried to use l'Hôpital's Rule on the following problem and got another indeterminate form. He is not sure what to do. Verify that Aram applied l'Hôpital's Rule correctly.

$$\lim\limits_{x \to 0} \frac{e^x - x - 1}{x^2} = \lim\limits_{x \to 0} \frac{e^x - 1}{2x}$$

b. Mara thinks she can help Aram by applying l'Hôpital's Rule again. Test Mara's theory and confirm your answer by graphing.

c. Use Mara's method to evaluate each of the following limits:

 i. $\lim\limits_{x \to \infty} \frac{x^2}{e^x}$

 ii. $\lim\limits_{x \to 0} \frac{1-\cos x}{x^2}$

5-163. Evaluate.

a. $\lim\limits_{x \to 0} (x^2 \cdot \csc^2 x)$

b. $\lim\limits_{x \to 0} \left(\frac{1}{\sin x} - \cot x \right)$

5-164. Find the error in the following solution. Then evaluate the limit correctly.

$$\lim\limits_{x \to 1} \frac{x^3 - x^2 + x - 1}{x^3 - x^2} = \lim\limits_{x \to 1} \frac{3x^2 - 2x + 1}{3x^2 - 2x} = \lim\limits_{x \to 1} \frac{6x - 2}{6x - 2} = 1$$

5-165. Your friend Jo-Jo is still unclear on the method used to find the limit in problem 5-164. She needs another example. Create your own indeterminate limit, then show the steps involved in finding the limit.

5-166. Find all values of k and m such that $\lim\limits_{x \to 0} \frac{k + \cos mx}{x^2} = -2$.

5-167. Determine the following limits. For each, describe your method. Homework Help ✎

   a. $\lim\limits_{x\to 1} \dfrac{\ln x}{x^2+1}$

   b. $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2-4}$

   c. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

   d. $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{\sin(x^2)}$

5-168. The formula $h(x) = (x-2)(ax-1)^2$ defines a family of functions, each corresponding to a different value of the parameter, a. Find the values of x for which each of these functions has a relative maximum or minimum; the answers will be in terms of a. Homework Help ✎

5-169. Find the antiderivative of each function. Homework Help ✎

   a. $f(x) = \sin x$

   b. $f(x) = \dfrac{2}{(x+3)^3}$

   c. $f(x) = \sec^2 x$

   d. $f(x) = \dfrac{3}{x^3} + \dfrac{2}{x^4}$

5-170. Find the area of the region enclosed by the curves $y = x^3 - 8x^2 + 20x$ and $y = -x^2 + 8x$.
Homework Help ✎

5-171. Find the derivative of each of the following functions. Homework Help ✎

   a. $f(x) = (2x+1)^2(3x-1)^3$

   b. $g(x) = \dfrac{1}{\sin(x^2)}$

   c. $f(x) = x\tan(x^2)$

   d. $g(x) = \cos(x\sqrt{x+1})$

5-172. A particle moves along a number line with velocity at time t given by the formula $v(t) = t\ln(t+1)$.

a. Use a 5-term midpoint Riemann sum to estimate the displacement of the particle over the interval from $t = 0$ to $t = 4$.

b. Estimate the same displacement using the Trapezoidal Rule with 5 trapezoids.

5-173. Use the definition of the derivative to write an expression (i.e. the limit of a difference quotient) representing each of the following derivatives. Then evaluate each limit to find the actual derivative. L'Hôpital's Rule may be used.

a. $f'(x)$, where $f(x) = \frac{1}{2x+1}$.

b. $h'(x)$, where $h(x) = (x - 2)^{-2}$.

c. $g'(4)$, where $g(x) = \sqrt{x}$.