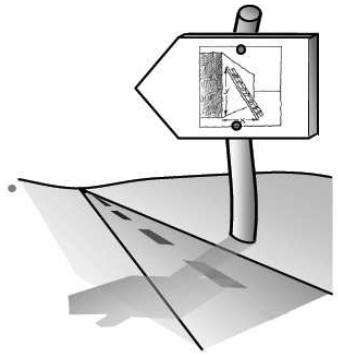
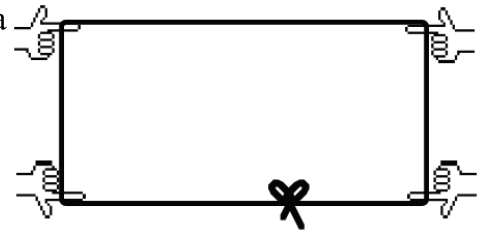


## 7.1.1 If I speed up, will you slow down?

### Related Rates Introduction



**7-1.** Using a closed loop of yarn, have two people in your team make a rectangle by holding two corners as shown in the picture. Start with your fingers together so that the length,  $l$ , is close to zero. Move your hands apart at a constant rate.



- As the length increases at a constant rate, how does the width change?
- If  $\frac{dw}{dt}$  is the rate the width is changing and  $\frac{dl}{dt}$  is the rate the length is changing, write an equation relating these two rates.
- As the length increases, how does the perimeter,  $P$ , change? What is  $\frac{dP}{dt}$ ?
- As the length increases, what happens to the area? Is the rate of change for the area a constant?

Changes in the length and width also affect other measurements, such as the area, perimeter, and the length of diagonals. These rates are not independent, since they depend on the rate of the changing dimensions. Because of this relationship, we call them **related rates**.

**7-2.** In each expression below:

$h$  = height

$f$  = number of fingers

$t$  = time

$H$  = length of hair if you never cut it

$a$  = your age

$T$  = number of teeth

Translate the rates below into a complete sentence, and determine whether the rates are constant, positive, negative, or some combination. For example,  $\frac{da}{dt}$  represents the rate at which your age changes over time (you could also say "compared to time," or "with respect to time" or "as time changes").

a.  $\frac{dh}{da}$

b.  $\frac{df}{dh}$

c.  $\frac{dH}{dt}$

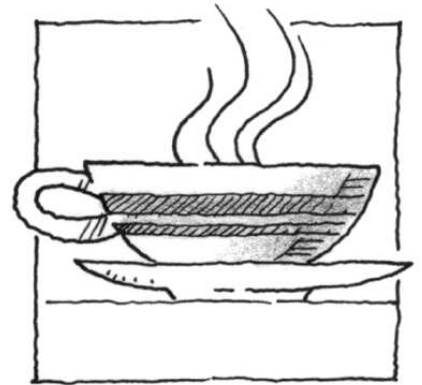
d.  $\frac{dT}{da}$

**7-3.** The amount of gas a car uses each hour depends on how fast the car is traveling. If  $x$  represents the distance a car travels and  $g$  represents the number of gallons of gas the car has consumed, analyze these related rates.

- What do  $\frac{dx}{dt}$  and  $\frac{dg}{dt}$  represent?
- Assume a car uses about 12 gallons of gas to travel 360 miles. What is  $\frac{dg}{dt}$  when the car is traveling 60 mph? 30 mph? 10 mph?
- Notice that  $\frac{dx}{dt}$  varies directly with  $\frac{dg}{dt}$ . Write an equation relating  $\frac{dx}{dt}$  and  $\frac{dg}{dt}$  for the car described in part (b) above.

**7-4.** Consider a coffee cup shaped like the one shown below. Suppose coffee is poured into the cup at a steady rate.

- Sketch a graph that shows  $h(t)$ , the height of coffee in the cup after  $t$  seconds. Is  $\frac{dh}{dt}$  positive, negative, or constant? How does  $\frac{dh}{dt}$  relate to the graph?
- Does volume vary directly with time? Sketch a graph that shows  $V(t)$ , the volume of coffee after  $t$  seconds. Is  $\frac{dV}{dt}$  positive, negative, or constant?
- Does the height of this coffee vary directly with time? Sketch a graph that shows  $h(t)$ , the height of the coffee after  $t$  seconds. Is  $\frac{dh}{dt}$  positive, negative, a constant?
- These last few problems are all focused on rates of change. Often, when something changes, several other measures change accordingly, causing their rates to be related. We call these **related rates**.



For the coffee cup, how is  $\frac{dh}{dt}$  related to  $\frac{dV}{dt}$ ? Discuss this with your team and write a complete description.

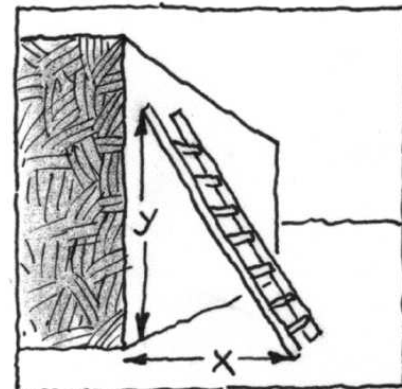
## 7-5. RELATED RATES

When two rates are related, we can describe their relationship with a **related rate statement**.

For example, as the population,  $p$ , of your school *increases* over time, the number of available lockers,  $l$ , *decreases*. That is, if  $\frac{dp}{dt}$  is positive,  $\frac{dl}{dt}$  is negative. If  $\frac{dp}{dt}$  is constant,  $\frac{dl}{dt}$  will be constant also. The faster  $\frac{dp}{dt}$  increases, the faster  $\frac{dl}{dt}$  decreases. The two rates are proportional, since each additional student decreases the availability of lockers by the same number.

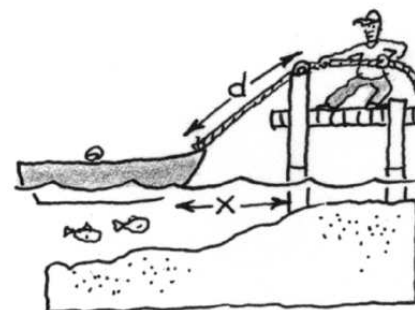
However, rates are not always proportional. Consider the situations below, and write related rate statements for the related rates in the scenarios below.

- a. A ladder is sliding down the side of a building. How does  $\frac{dy}{dt}$  relate to  $\frac{dx}{dt}$ ?



- i. What do  $x$  and  $y$  represent?
- ii. What do  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  represent?
- iii. If Katie was standing on the top of the ladder and Sarah began pulling the bottom of the ladder at a constant rate away from the wall, would  $\frac{dx}{dt}$  be positive or negative? How about  $\frac{dy}{dt}$ ? Explain.
- iv. As Sarah pulls the ladder away from the wall at a constant rate, describe the way  $y$  changes.
- v.  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  are not directly proportional. Does this mean that they are inversely proportional?

- b. As a man pulls a rope (tied to a boat), the boat floats toward the dock. What do  $x$  and  $d$  represent? What do  $\frac{dx}{dt}$  and  $\frac{dd}{dt}$  represent? How are  $\frac{dx}{dt}$  and  $\frac{dd}{dt}$  related?



Also, how does  $\frac{dd}{dt}$  relate to the rate at which the man pulls the rope?



**7-6.** Differentiate each equation with respect to  $x$ . Leave your answers in terms of only  $y$  and  $x$ .

[Homework Help](#)

a.  $y = 7\ln(x + 1) - x^2$

b.  $2^x + 2^y = e$

c.  $y = e^{\tan(x)}$

d.  $(5x + 1)^2 + (y + 1)^2 = 1$

e. Find  $\frac{d^2y}{dx^2}$  for part (a). Be sure to leave all parts in terms of  $y$  and  $x$  only.

f. Find  $\left. \frac{d^2y}{dx^2} \right|_{x=3}$  for part (a).

g. Write the equation of the tangent line at  $x = 3$  for part (a).


h. Is the tangent line an over or under approximation? Use part (f) to determine your answer.

7-7. Integrate. [Homework Help](#) 

a.  $\int e^{3x} dx$


b.  $\int \sin(3x) dx$

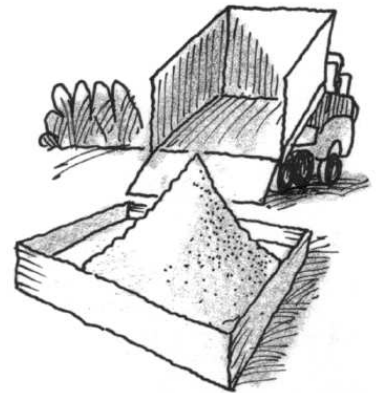
c.  $\int [(3x+1)^3 + 3x + 3] dx$


7-8. For each function below, find its average value on the given interval and state the time  $t$  at which it equals its average value. [Homework Help](#) 


a.  $g(t) = 3t + 6$  for  $[0, 8]$

b.  $g(t) = 3e^t$  for  $[0, 1]$

7-9. Sand is being poured into a sandbox, making a conical pile with radius  $r$  and volume  $V$ . Explain what  $\frac{dV}{dt}$  and  $\frac{dr}{dV}$  represent. [Homework Help](#) 



7-10. Sketch the region bounded by  $y = x^4$  and  $y = 1$ . Find the area of the region *exactly*. [Homework Help](#) 

7-11. Find two positive numbers whose sum is 8, such that the sum of the square of the first and the cube of the second is a minimum. Use calculus to justify your solution. [Homework Help](#) 

7-12. As the length,  $k$ , of rope changes, the kite flies horizontally either toward the person or away from the person. (The altitude of the kite remains constant). Are  $\frac{dk}{dt}$  and  $\frac{dx}{dt}$  positive or negative? Describe the way  $\frac{dx}{dt}$  changes. [Homework Help](#) 