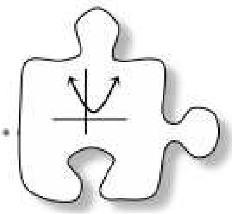
8.2.3 Where are the complex numbers?



More Complex Numbers and Equations

If the number line is filled with real numbers, how can imaginary and complex numbers be represented geometrically? In this lesson, you will learn a way to graph complex numbers and interpret the graphical meaning of complex solutions.

8-97. Avi and Tran were trying to figure out how they could represent complex numbers geometrically. Avi decided to make a number line horizontal like the x-axis to represent the real part as well as a vertical line like the y-axis to represent the imaginary part.

a. Draw a set of axes and label them as Avi described.

b. How could Avi and Tran graph a point to represent the complex number $3 + 4i$? Be prepared to share your strategies with the class.

c. Use the method from part (b) to plot points to represent the six numbers below.

i. $2 + 5i$

ii. $6 - i$

iii. $-5 - 3i$

iv. 4

v. $7i$

vi. $-4 + 2i$

8-98. On a new set of **complex axes** (like those drawn by Avi and Tran in problem 8-97), locate points representing all of the following complex numbers.

a. $3 + 4i$, $3 - 4i$, $-3 + 4i$, and $-3 - 4i$.

b. The four complex numbers represented by $\pm 4 \pm 3i$.

c. 5, -5, $5i$, and $-5i$

d. What do you notice about your graph? How far from (0, 0) is each point?

8-99. On the real number line, the distance from 0 to a point on the line is defined as the absolute value of the number. Similarly, in the **complex plane** (the plane defined by a set of complex axes), the **absolute value**of a complex number is its distance from zero or the origin (0, 0). In the previous problem, the absolute value of all of those complex numbers was 5. For each of the following questions, a sketch in the complex plane will help in visualizing the result.

a. What is the absolute value of $-8 + 6i$?

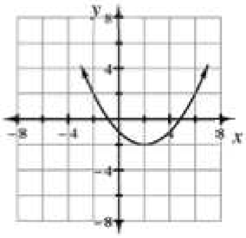
b. What is the absolute value of $7 - 2i$?

c. What is $|4 + i|$?

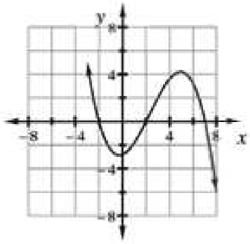
d. What is the absolute value of $a + bi$?

8-100. Based on the following graphs, how many *real*roots does each polynomial function have?

a.

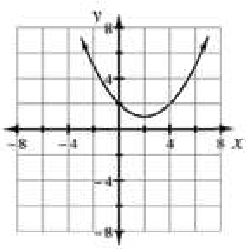


b.

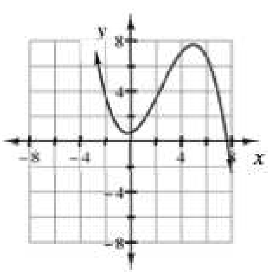


Graphs (a) and (b) above have been vertically shifted to create graphs (c) and (d) shown below. How many *real* roots does each of these new polynomial functions have?

a.

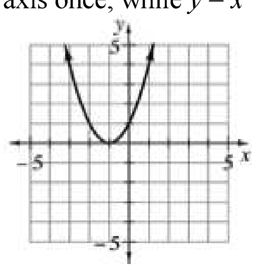


b.

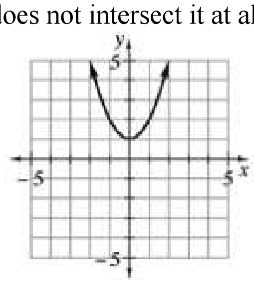


The polynomials in parts (c) and (d) do not have fewer roots. Polynomial (c) still has *two* roots, but now the roots are complex. Polynomial (d) has *three* roots: two are complex, and only one is real.

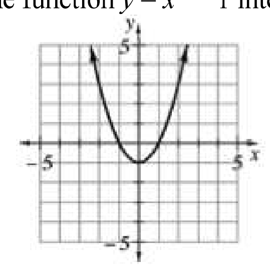
8-101. Recall that a polynomial function with degree *n* crosses the *x*-axis at most *n* times. For instance, $y = (x + 1)^2$ intersects the *x*-axis once, while $y = x^2 + 1$ does not intersect it at all. The function $y = x^2 - 1$ intersects it twice. These graphs are shown below.



$y = (x+1)^2$



$y = x^2 + 1$



$y = x^2 - 1$

a. The graph of a third-degree equation might intersect the *x*-axis one, two, or three times. Make sketches of all these possibilities.

b. Can a third-degree equation have zero real roots? Explain why or why not.

8-102. Now consider the graph of $y = x^3 - 3x^2 + 3x - 2$.

a. How many real solutions could $x^3 - 3x^2 + 3x - 2 = 0$ have?

b. Check to verify that $x^3 - 3x^2 + 3x - 2 = (x - 2)(x^2 - x + 1)$.

c. Find all of the solutions of $x^3 - 3x^2 + 3x - 2 = 0$.

d. How many x-intercepts does $y = x^3 - 3x^2 + 3x - 2$ have? How many real roots and how many non-real roots (complex)?

8-103. Sketch the graph of $f(x) = x^2 + 4$ and solve the equation $x^2 + 4 = 0$ to find its roots.

a. Describe the parabola. Be sure to include the vertex and the equation of its axis of symmetry.

b. With a partner, obtain a copy of the Lesson 8.2.3 Resource Page from your teacher, and follow the directions below to make a 3-D model that will show the location of the complex roots in a complex plane that is perpendicular to the real plane in which you drew the graph of the parabola.

i. Fold the paper on the line marked ***bi*** and ***−bi.*** This is a "mountain" fold, so the printing is on the outside.

ii. Cut the paper exactly along the dotted line. Do not cut beyond the dotted portion.

iii. Now make "valley" folds on the two lines parallel to the first fold.

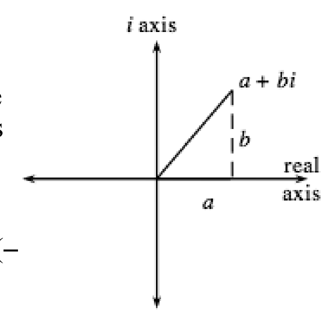
iv. Hold the two ends of the paper and push them toward the center so the center pops up, and then fold the top and bottom of the paper back on the line marked "center."

You should have a three-dimensional coordinate system with the xy-plane facing you and the i-axis coming out toward you. The equation $f(x) = x^2 + 4$ should be in the lower right corner. Now locate the roots of the function on your 3-D model.

METHODS AND MEANINGS

MATH NOTES

Graphing Complex Numbers

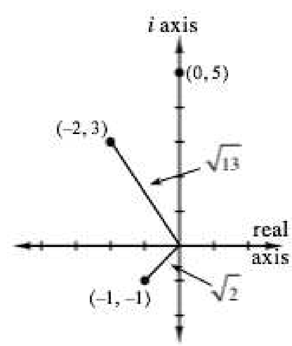


To represent complex numbers, an imaginary axis and a real axis are needed. Real numbers are on the horizontal axis and imaginary numbers are on the vertical axis, as shown in the examples below. This representation is called the **complex plane**.

In the complex plane, $a + bi$ is located at the point (a, b).The number $2 + 3i$ is located at the point (2, 3). The number i or $0 + 1i$ is located at (0, 1). The number –2 or $-2 + 0i$ is located at (–2, 0).

The **absolute value** of a complex number is its distance from the origin. To find the absolute value, calculate the distance from (0, 0) to (a,b):

$|a+bi| = \sqrt{a^2+b^2}$



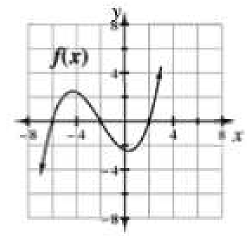
Examples:

$|-2+3i| = \sqrt{(-2)^2+3^2} = \sqrt{13}$

$|-1-i| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$

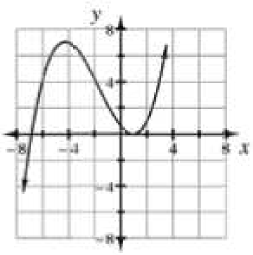
$|5i| = 5$

8-104. In parts (a) through (d) below, for each polynomial function $f(x)$, the graph of $f(x)$ is shown. Based on this information, state the number of linear and quadratic factors the factored form of its equation should have and how many real and complex (non-real) solutions $f(x) = 0$ might have. (Assume a polynomial function of the lowest possible degree for each one.)

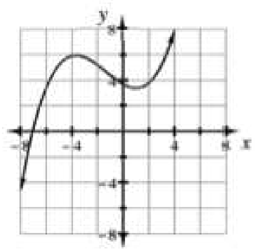


Example: $f(x)$ at right will have three linear factors, therefore three real roots and no complex roots.
Homework Help ✎

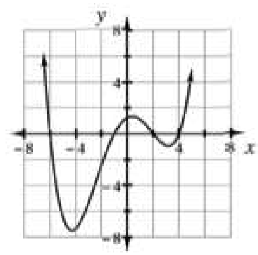
a.



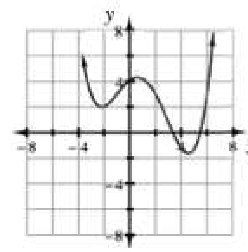
b.



c.



d.



8-105. Make a sketch of a graph $p(x)$ so that $p(x) = 0$ would have the indicated number and type of solutions. Homework Help ✎

a. 5 real solutions

b. 3 real and 2 complex

c. 4 complex

d. 4 complex and 2 real

e. For parts (a) through (d), what is the lowest degree each function could have?

8-106. Consider the function $y = x^3 - 9x$. Homework Help ✎

a. What are the roots of the function? (Factoring will help!)

b. Sketch a graph of the function.

8-107. Make rough sketches of the graphs of each of the following polynomial functions. Be sure to label the x- and y- intercepts.
Homework Help ✎

a. $y = x(2x + 5)(2x - 7)$

b. $y = (15 - 2x)^2(x + 3)$

8-108. Graph $y \geq |x + 2| - 3$ and $y \leq 2$ on the same set of axes. Homework Help ✎

8-109. Fireworks for the annual Fourth of July show are launched straight up from a steel platform. The launch of the entire show is computer controlled. The height of a particular firework in meters off ground level is given by $h = -4.9t^2 + 49t + 11.27$, where time, t, is in seconds. Homework Help ✎

a. What was the height of the platform? What is the maximum height the firework reached? How many seconds until it hit the ground?

b. Rewrite the equation in factored form. Why might factored form of the equation be useful?

8-110. You are given the equation $5x^2 + bx + 20 = 0$. For what values of b does this equation have real solutions? Homework Help ✎

8-111. Show that each of the following equations is true. Homework Help ✎

a. $(i - 3)^2 = 8 - 6i$

b. $(2i - 1)(3i + 1) = -7 - i$

c. $(3 - 2i)(2i + 3) = 13$

8-112. Consider the functions $y = \frac{1}{2}$ and $y = \frac{16}{x^2 - 4}$. Find the coordinates where the graphs of the functions intersect. Homework Help ✎