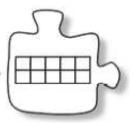
8.3.2 How can I solve it?

Factors and Integral Roots



You already know several methods for solving quadratic equations, such as factoring, completing the square, and using the Quadratic Formula. Mathematicians have developed formulas for solving third- and fourth-degree polynomial equations, but these formulas are far more complicated and messy than the Quadratic Formula is, and they are rarely used. Furthermore, for polynomials of degree greater than four, there is no single formula to use. For many polynomials, you can develop more useful methods than a formula based on what you already know.

8-129. SEARCHING FOR ROOTS OF POLYNOMIALS

By combining what you know about graphing, factoring, polynomial division, and solving quadratic equations, you will be able to find the roots of the higher-degree polynomial functions in this lesson.

Your Task: Find all of the zeros of the polynomial below and then write the polynomial in factored form with factors of degree 2 or 1.

$$P(x) = x^4 - x^3 - 5x^2 + 3x + 6$$

Discussion Points

What are some possible linear factors?

How can the graph help us decide which factors to try?

How can we use the known factors to figure out other factors?

What do we need to do to write the polynomial in factored form and find the zeros?

Further Guidance

8-130. Given the polynomial $P(x) = x^4 - x^3 - 5x^2 + 3x + 6$, list all of the possible linear factors $(x \pm a)$ of the polynomial. Use only integer values for a.

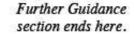
- a. Could x 5 be included on your list of possible linear factors? Explain.
- b. Not all of the *possibilities* will actually be factors. Use your graphing calculator to decide which of the factors you listed are the best possibilities.
- c. Now that you have shortened the list of possibilities, which factors on the shortened list really are factors of P(x)? Justify your answer.



d. If you have not already, divide the polynomial by one of the factors from part (c) and write the

polynomial as a product of a linear and a cubic factor.

- e. Now divide the cubic factor from part (d) by the other linear factor that worked, and write the original polynomial as a product of two linear factors and one quadratic factor.
- f. From the factored form, you can find all of the solutions to $x^4 x^3 5x^2 + 3x + 6 = 0$ and the exact x-intercepts for the graph of P(x). What are they?



8-131. LEARNING LOG

As a team, look back over the work you did to find the linear factors of P(x) and make a list of steps you can use to find all of the zeros of a given polynomial. Then record your ideas in your Learning Log. Use diagrams, arrows, and other math tools to help demonstrate your ideas. Title this entry "Factors and Roots of Polynomial Functions" and label it with today's date.



8-132. Your teacher will assign your team one of the following polynomial functions. Use the list of steps your team developed to factor the polynomial and find all of its roots. Then prepare a poster in which you illustrate and justify each of your steps. Be sure to include the graph on your poster and clearly explain the relationship between the solutions of the equation and the *x*-intercepts.

a.
$$Q_1(x) = x^3 + 3x^2 + 1x - 5$$

b.
$$Q_2(x) = 6x^4 + 7x^3 - 36x^2 - 7x + 6$$

c.
$$Q_3(x) = x^4 + 2x^3 + 10x^2 + 18x + 9$$

d.
$$Q_4(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$$

e.
$$Q_5(x) = x^5 - 4x^3 - x^2 + 4$$

f.
$$Q_6(x) = x^3 + x^2 - 7x - 7$$

8-133. LEARNING LOG

Make additions and/or adjustments to your Learning Log entry from problem 8-131 to reflect what you leaned from the posters that you and your classmates presented for problem 8-132.

8-134. Use the procedures you developed to factor each of the following polynomial expressions. Each final answer should be a linear factor times a quadratic factor. Look for patterns in the factors.

a.
$$x^3 - 1$$

b.
$$x^3 + 8$$

c.
$$x^3 - 27$$

d.
$$x^3 + 125$$

8-135. If you generalize the patterns for the factors in problem 8-134, you will discover a shortcut for factoring cubic polynomials that are described as the sum or difference of two cubes. Write the factors for each polynomial expression below.

a.
$$x^3 + a^3$$

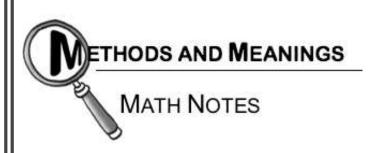
b.
$$x^3 - b^3$$

8-136. Are there similar patterns for $x^4 + a^4$ and $x^4 - b^4$? Explain.

8-137. BUILDING POLYNOMIALS

For each of the following descriptions of polynomial functions with integral coefficients, answer each question below.

- i. What are the possible numbers of real zeros?
- ii. How many complex zeros are possible?
- iii. For each number of possible real zeros, give an example of a polynomial in factored form.
- a. A third-degree polynomial function.
- b. A fourth-degree polynomial function.
- c. A fifth-degree polynomial function.



Polynomial Theorems

Factor Theorem: If x = a is a zero of a polynomial function, then p(x) then (x - a) is a factor of the polynomial, and if (x - a) is a factor of the polynomial, then x = a is a zero of the polynomial.

Example: If x = -3, x = 2, and $x = 1 \pm i\sqrt{7}$ are zeros of the polynomial $p(x) = 2x^4 - x^3 + 20x - 48$. According to the Factor Theorem, $(x + 3), (x - 2), (x - (1 + i\sqrt{7}))$, and $(x - (1 - i\sqrt{7}))$, are factors of the polynomial.

Fundamental Theorem of Algebra: The Fundamental Theorem of Algebra states that every one-variable polynomial has at least one complex root (remember that every real number is also a complex number). This theorem can be used to show that every polynomial of degree n has n complex roots. This also means that the polynomial has n linear factors, since for every root a, (x - a) is a linear factor.

Note that the total of n roots might include roots that occur multiple times. For example, the third degree polynomial $p(x) = x^3 - 3x^2 + 4$ has three roots—a root at x = -1 and a double root at x = 2. The three linear factors are (x + 1), (x - 2), and (x - 2)

Integral Zero Theorem: For any polynomial with integral coefficients, if an integer is a zero of the polynomial, it must be a factor of the constant term.

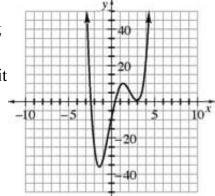
Example: Suppose the integers a, b, c, and d are zeros of a polynomial. Then, according to the Factor Theorem, (x - a)(x - b)(x - c)(x - d) are factors of the polynomial. When you multiply these factors together, the constant term will be abcd, so a, b, c, and d are factors of the constant term.

Remainder Theorem: For any number c, when a polynomial p(x) is divided by (x - c), the remainder is p(c). For example, if the polynomial $p(x) = x^4 - x^3 + 20x$ —48 is divided by (x - 5), the remainder is p(5) = 552.

Note that it follows from the Remainder Theorem that if p(c) = 0, then (x - c), is a factor. For example, if $p(x) = x^3 - 3x^2 + 4$, one solution to $x^3 - 3x^2 + 4 = 0$ is x = -1. Since p(-1) = 0, then (x + 1) is a factor.



8-138. Carlo was trying to factor the polynomial $p(x) = x^4 - 4x^3 - 4x^2 + 24x - 9$ to find all of its roots. He had already found one factor by making a guess and dividing the polynomial, so he had $p(x) = (x - 3)(x^3 - x^2 - 7x + 3)$. He was trying to factor $x^3 - x^2 - 7x + 3$, so he had tried dividing it by (x + 3), (x + 1), and (x - 1), but none worked without a remainder. Then Teo came by and said, "You should look at the graph." Homework Help



- a. How does the graph help?
- b. Complete the problem.
- **8-139.** Spud has a problem. He knows that the solutions for a quadratic equation are x = 3 + 4i and x = 3 4i

4i, but in order to get credit for the problem he was supposed to have written down the original equation. Unfortunately, he lost the paper with the original equation on it. Luckily, his friends are full of advice. Homework Help

- a. Alexia says, "Just remember when we made polynomials. If you wanted 7 and 4 to be the answers, you just used (x-7)(x-4). So you just do x minus the first one times x minus the other." Use (x-(3+4i))(x-(3-4i)) to find the quadratic expression.
- b. Hugo says, "No, no, no. You can do it that way, but that's too complicated. I think you just start with x = 3 + 4i and work backward. So x 3 = 4i, then, hmmm. Yeah, that'll work." Try Hugo's method.
- c. Whose method do you think Spud should use? Explain your choice.
- **8-140.** So far you have been able to extend the rules for real numbers to add, subtract, and multiply complex numbers, but what about dividing? Can you use what you know about real numbers to divide one complex number by another? In other words, if a problem looks like this:

$$\frac{3+2i}{-4+7i}$$

What needs to be done to get an answer in the form of a single complex number, a + bi?

Natalio had an idea. He said, "I'll bet we can use the conjugate!"

"How?" asked Ricki.

"Well, it's a fraction. Can't we multiply the numerator and denominator by the same number?" Natalio replied. Homework Help ▶

- a. Natalio was not very clear in his explanation. Show Ricki what he meant they should do.
- b. Using Natalio's ideas you probably still came up with a fraction in part (a), but the denominator should be a whole number. To write a complex number such as $\frac{c+di}{m}$ in the form a+bi, just use the distributive property to rewrite the result as $\frac{c}{m} + \frac{d}{m}i$. Rewrite your result for part (a) in this form.
- **8-141.** Use the method developed in problem 8-140 to do the following division problems. <u>Homework Help </u> **S**

a.
$$\frac{2-5i}{1-2i}$$

b.
$$(-3+i) \div (2+3i)$$

- **8-142.** Find the inverse of $g(x) = (x+1)^2 3$ with the domain $x \ge -1$. Sketch both graphs and state the domain and range of the inverse function. Homework Help
- **8-143.** Sketch the graph of each polynomial function below and find all of the zeros. Homework Help

a.
$$v = x^3 + 1$$

b.
$$y = x^3 - 8$$

8-144. Solve the system of equations below for (x, y, z). Homework Help

$$x = y + z$$

$$2x + 3y + z = 17$$

$$z + 2y = 7$$

8-145. Solve
$$\sqrt{x^2+6} = x+2$$
. Homework Help

8-146. Sketch the graph of each equation below. Homework Help **№**

a.
$$y = 3 \sin(x + \frac{\pi}{2})$$

b.
$$y = -2 \sin(4x)$$

8-147. Spud has done it again. He's lost another polynomial function. This one was a cubic, written in standard form. He knows that there were two complex zeros, $-2 \pm 5i$ and one real zero, -1. What could his original function have been? Homework Help

8-148. Given the polynomial $p(x) = x^3 - 6x^2 + 7x + 2$. Homework Help

- a. What is p(2)?
- b. Use the Remainder Theorem to find one factor of $x^3 6x^2 + 7x + 2$. (See the Math Notes box in Lesson 8.3.2 above.)
- c. Use (b) to find another factor.
- d. What are all the solutions of $x^3 6x^2 + 7x + 2 = 0$?

8-149. Rewrite each of the following division problems as a single complex number. Homework Help **№**

a.
$$\frac{5+3i}{5-3i}$$

b.
$$\frac{7+3i}{1-i}$$

8-150. Megan is an industrial engineer for Bowler Cola Company. She takes a random sample of cola cans from the production line each day to determine if the product meets various specifications. One of the measurements she records is the mass (in grams) of the filled cans. The following sorted data are from of a sample of 30 regular and diet cola cans.

Note: The data is sorted so it is easy to work with without a statistical calculator. Homework Help №

Regula	r									
361	362	363	365	366	366	367	367	367	368	
368	368	369	369	369	369	370	370	370	371	
370	371	371	371	373	375	375	376	376	380	
							Checksum 11088			
Diet										
349	349	350	351	353	353	353	354	354	354	
354	355	355	355	356	357	358	361	361	361	
361	361	361	362	362	363	364	365	366	366	
	Checksum 1072									

- a. Find the five number summary (minimum, third quartile, median, first quartile, maximum) for each soda.
- b. Make a combination histogram and boxplot for each type of soda. Include the five number summary. Use an interval of 348 to 384 grams on the *x*-axis and a bin width of 4 grams.
- c. Describe the center, shape, spread, and any outliers, of each histogram.
- d. Compare the two samples.
- e. Each can is marked as containing 12 fluid ounces. Twelve ounces is about 341 grams. Why is there so much variation from 341 grams in the samples?
- **8-151.** Sketch both the circle $x^2 + y^2 = 25$ and the parabola $y = x^2 13$. Homework Help
 - a. How many points of intersection are there?
 - b. Find the coordinates of these points algebraically.
- **8-152.** Solve each equation. Be sure to check your answers. <u>Homework Help </u>

a.
$$\sqrt{x} + 2 = x$$

b
$$\sqrt{x} + 2 = \sqrt{x+6}$$

8-153. Sketch a graph of each of the following trigonometric functions. <u>Homework Help №</u>

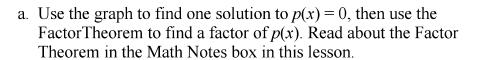
a.
$$y = \sin(0.5(x - \pi))$$

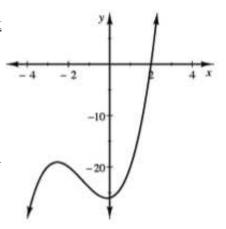
b.
$$y = 10\sin(3x) - 2$$

c.
$$y = 5\cos(x + \frac{\pi}{4})$$

d.
$$y = \cos(2(x - \frac{\pi}{4}))$$

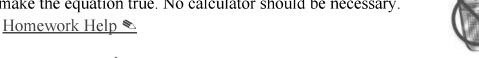
8-154. The graph of $p(x) = x^3 + 4x^2 + x - 26$ is shown at right. Help





b. Determine all the real and complex zeros of p(x). Then write p(x) in factored form.

8-155. For each equation, find two solutions $0 \le x < 2\pi$, which make the equation true. No calculator should be necessary.



a.
$$\cos x = -\frac{1}{2}$$

b.
$$\tan x = \frac{\sqrt{3}}{3}$$

c.
$$\sin x = 0$$

d.
$$\cos x = \frac{\sqrt{2}}{2}$$

8-156. Rewrite each of the following division problems as a single complex number in simplest form. Homework Help

a.
$$\frac{2-6i}{4+2i}$$

b.
$$\frac{5}{1+2i}$$

8-157. A long lost relative died and left you \$15,000! Your parents say that you need to save the money for college, so you put it an account that pays 8% interest compounded annually. How many years will it take until your account is worth \$25,000? Homework Help

8-158. Solve each equation. <u>Homework Help </u> **Solve**

a.
$$\log_3(2x-1) = -2$$

b.
$$5^{\log_5(x)} = 3$$

c.
$$\log_2(x) - \log_2(3) = 4$$

d.
$$\log_3(5) = x$$

8-159. Verify that the graphs of the equations $x^2 + y^3 = 17$ and $x^4 - 4y^2 - 8xy = 17$ intersect at (3, 2). Homework Help

8-160. The graphs of $y = \log_2(x - 1)$ and $y = x^3 - 4x$ intersect at two points: (2, 0) and approximately (1.1187, -3.075). Use that information to solve $\log_2(x - 1) = x^3 - 4x$. Homework Help

8-161. Solve each equation below for x. Homework Help extstyle extst

a.
$$1234x + 23456 = 987654$$

b.
$$\frac{10}{x} + \frac{20}{x} = 5$$

c.
$$5x^2 - 6x + 1 = 0$$

d.
$$x^3 - 3x^2 + 2x = 0$$

8-162. Use your knowledge of the unit circle to explain why the graphs of $y = \sin\theta$ and $y = \cos(\theta - \frac{\pi}{2})$ are the same. Homework Help

8-163. For homework, Londa was asked to simplify the expression $\sqrt{-7} \cdot \sqrt{-7}$. She got the answer 7, but when she checked, she learned that the correct answer was -7. Homework Help



- a. Show Londa the steps she could take to get -7.
- b. What steps do you think Londa took to get 7 as a result?
- c. What does she need to consider to avoid making this mistake in the future?
- d. Londa's example means that it is not always true that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ for real numbers a and b. What restriction needs to be placed on the numbers a and b?

8-164. Change each angle from degrees to radians. Homework Help **№**

- a. 60°
- b. 75°
- c. 210°
- d. 225°