Core Connections Algebra 2
Checkpoint Materials

Note to Students (and their Teachers)

Students master different skills at different speeds. No two students learn exactly the same way at the same time. At some point you will be expected to perform certain skills accurately. Most of the Checkpoint problems incorporate skills that you should have developed in previous courses. If you have not mastered these skills yet it does not mean that you will not be successful in this class. However, you may need to do some work outside of class to get caught up on them.

Starting in Chapter 2 and finishing in Chapter 12, there are 18 problems designed as Checkpoint problems. Each one is marked with an icon like the one above. After you do each of the Checkpoint problems, check your answers by referring to this section. If your answers are incorrect, you may need some extra practice to develop that skill. The practice sets are keyed to each of the Checkpoint problems in the textbook. Each has the topic clearly labeled, followed by the answers to the corresponding Checkpoint problem and then some completed examples. Next, the complete solution to the Checkpoint problem from the text is given, and there are more problems for you to practice with answers included.

Remember, looking is not the same as doing! You will never become good at any sport by just watching it, and in the same way, reading through the worked examples and understanding the steps is not the same as being able to do the problems yourself. How many of the extra practice problems do you need to try? That is really up to you. Remember that your goal is to be able to do similar problems on your own confidently and accurately. This is your responsibility. You should not expect your teacher to spend time in class going over the solutions to the Checkpoint problem sets. If you are not confident after reading the examples and trying the problems, you should get help outside of class time or talk to your teacher about working with a tutor.

Checkpoint Topics

Checkpoint 2A

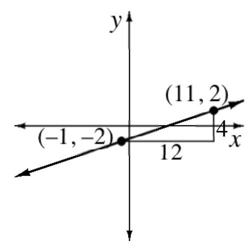
Problem 2-53

Finding the Distance Between Two Points and the Equation of a Line

Answers to problem 2-53: a: $\sqrt{45} = 3\sqrt{5} \approx 6.71$; $y = \frac{1}{2}x + 5$, b: 5; $x = 3$, c: $\sqrt{725} \approx 26.93$; $y = -\frac{5}{2}x + \frac{5}{2}$, d: 4; $y = -2$

The distance between two points is found by using the Pythagorean Theorem. The most commonly used equation of a line is $y = mx + b$ where m represents the slope of the line and b represents the y-intercept of the line. One strategy for both types of problems is to create a generic right triangle determined by the given points. The lengths of the legs of the triangle are used in the Pythagorean Theorem to find the distance. They are also used in the slope ratio to write an equation of the line. This strategy is not necessary for vertical or horizontal pairs of points, however.

Example: For the points $(-1,-2)$ and $(11,2)$, find the distance between them and determine an equation of the line through them.



Solution: Using a generic right triangle, the legs of the triangle are 12 and 4. The distance between the points is the length of the hypotenuse.

$$d^2 = 12^2 + 4^2 = 160 \Rightarrow d = \sqrt{160} = 4\sqrt{10} \approx 12.65$$

The slope of the line, $m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{4}{12} = \frac{1}{3}$. Substituting this into the equation of a line, $y = mx + b$, gives $y = \frac{1}{3}x + b$. Next substitute any point that is on the line for x and y and solve for b. Using $(11,2)$, $2 = \frac{1}{3} \cdot 11 + b$, $2 = \frac{11}{3} + b$, $b = -\frac{5}{3}$.

The equation is $y = \frac{1}{3}x - \frac{5}{3}$.

Some people prefer to use formulas that represent the generic right triangle.

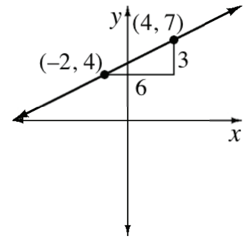
$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-2)}{11-(-1)} = \frac{4}{12} = \frac{1}{3}$$

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(11-(-1))^2 + (2-(-2))^2} = \sqrt{12^2 + 4^2} = \sqrt{160}$$

Notice that $x_2 - x_1$ and $y_2 - y_1$ represent the lengths of the horizontal and vertical legs respectively.

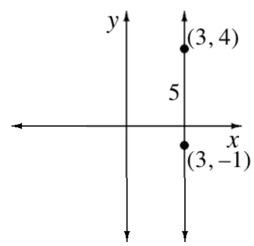
Now we can go back and solve the original problems.

a. $d^2 = 6^2 + 3^2 \Rightarrow d^2 = 45 \Rightarrow d = \sqrt{45} = 3\sqrt{5} \approx 6.71$



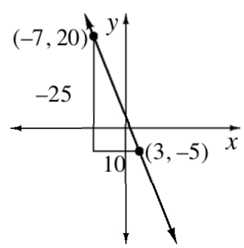
$m = \frac{3}{6} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + b$
Using the point $(4, 7) \Rightarrow 7 = \frac{1}{2} \cdot 4 + b \Rightarrow b = 5$.
The equation is $y = \frac{1}{2}x + 5$.

b. Since this is a vertical line, the distance is simply the difference of the y values. $d = 4 - (-1) = 5$.



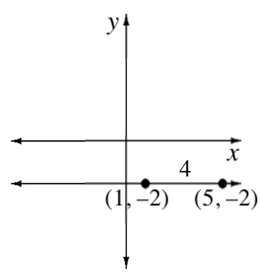
Vertical lines have an *undefined* slope and the equation of the line is of the form $x = k \Rightarrow x = 3$.

c. $d^2 = (-25)^2 + 10^2 \Rightarrow d^2 = 725 \Rightarrow d = \sqrt{725} \approx 26.93$



$m = \frac{-25}{10} = -\frac{5}{2} \Rightarrow y = -\frac{5}{2}x + b$
Using the point $(3, -5) \Rightarrow -5 = -\frac{5}{2} \cdot 3 + b \Rightarrow b = -5 + \frac{15}{2} = \frac{5}{2}$
The equation is $y = -\frac{5}{2}x + \frac{5}{2}$.

d. Since this is a horizontal line, the distance is simply the difference of the x-values. $d = 5 - 1 = 4$.



Horizontal lines have a slope of 0 and the equation of the line is of the form $y = k \Rightarrow y = -2$.

Here are some more to try. For each pair of points, compute the distance between them and then find an equation of the line through them.

1. $(2,3)$ and $(1,2)$
2. $(-3,-5)$ and $(-1,0)$
3. $(4,2)$ and $(8,-1)$
4. $(1,3)$ and $(5,7)$
5. $(0,4)$ and $(-1,-5)$
6. $(-3,2)$ and $(2,-3)$
7. $(4,2)$ and $(-1,-2)$
8. $(3,1)$ and $(-2,-4)$
9. $(4,1)$ and $(4,10)$
10. $(10,2)$ and $(2,22)$
11. $(-10,3)$ and $(-2,-5)$
12. $(-3,5)$ and $(12,5)$
13. $(-4,10)$ and $(-6,15)$
14. $(-6,-3)$ and $(2,10)$

Answers:

1. $\sqrt{2} \approx 1.41$; $y = x + 1$
2. $\sqrt{29} \approx 5.39$; $y = \frac{5}{2}x + \frac{5}{2}$
3. 5; $y = -\frac{3}{4}x + 5$
4. $\sqrt{32} = 4\sqrt{2} \approx 5.66$; $y = x + 2$
5. $\sqrt{82} \approx 9.06$; $y = 9x + 4$
6. $\sqrt{50} = 5\sqrt{2} \approx 7.07$; $y = -x - 1$
7. $\sqrt{41} \approx 6.40$; $y = \frac{4}{5}x - \frac{6}{5}$
8. $\sqrt{50} = 5\sqrt{2} \approx 7.07$; $y = x - 2$
9. 9; $x = 4$
10. $\sqrt{464} \approx 21.54$; $y = -\frac{5}{2}x + 27$
11. $\sqrt{128} = 8\sqrt{2} \approx 11.31$; $y = -x - 7$
12. 15; $y = 5$
13. $\sqrt{29} \approx 5.39$; $y = -\frac{5}{2}x$
14. $\sqrt{233} \approx 15.26$; $y = \frac{13}{8}x + \frac{27}{4}$

Checkpoint 2B

Problem 2-152

Solving Linear Systems in Two Variables

Answers to problem 2-152: (3, 2)

You can solve systems of equations using a variety of methods. For linear systems, you can graph the equations, use the Substitution Method, or use the Elimination Method. Each method works best with certain forms of equations. Following are some examples. Although the method that is easiest for one person may not be easiest for another, the most common methods are shown below.

Example 1: Solve the system of equations $x = 4y - 7$ and $3x - 2y = 1$.

Solution: For this, we will use the Substitution Method. Since the first equation tells us that x is equal to $4y - 7$, we can substitute $4y - 7$ for x in the second equation. This allows us to solve for y, as shown at right.

$$\begin{aligned} 3(4y-7)-2y&=1\\ 12y-21-2y&=1\\ 10y-21&=1\\ 10y&=22\\ y&=\tfrac{22}{10}=2.2 \end{aligned}$$

Then substitute $y = 2.2$ into either original equation and solve for x: Choosing the first equation, we get $x = 4(2.2) - 7 = 8.8 - 7 = 1.8$. To verify the solution completely check this answer in the second equation by substituting. $3(1.8) - 2(2.2) = 5.4 - 4.4 = 1$

Answer: The solution to the system is $x = 1.8$ and $y = 2.2$ or $(1.8, 2.2)$.

Example 2: Solve the system of equations $y = \frac{3}{4}x - 1$ and $y = -\frac{1}{3}x - 1$.

Solution: Generally graphing the equations is not the most efficient way to solve a system of linear equations. In this case, however, both equations are written in y = form so we can see that they have the same y-intercept. Since lines can cross only at one point, no points or infinite points, and these lines have different slopes (they are not parallel or coincident), the y-intercept must be the only point of intersection and thus the solution to the system. We did not actually graph here, but we used the principles of graphs to solve the system. Substitution would work nicely as well.

Answer: $(0, -1)$

Example 3: Solve the system $x+2y=16$ and $x-y=2$.

Solution: For this, we will use the Elimination Method. We can subtract the second equation from the first and then solve for y, as shown at right.

$$\begin{array}{r} x+2y=16 \\ -(x-y=2) \\ \hline 0+3y=14 \\ 3y=14 \\ y=\frac{14}{3} \end{array}$$

We then substitute $y=\frac{14}{3}$ into either original equation and solve for x. Choosing the second equation, we get $x-\frac{14}{3}=2$, so $x=2+\frac{14}{3}=\frac{20}{3}$. Checking our solution can be done by substituting both values into the first equation.

Answer: The solution to the system is $(\frac{20}{3},\frac{14}{3})$.

Example 4: Solve the system $x+3y=4$ and $3x-y=2$.

Solution: For this, we will use the Elimination Method, only we will need to do some multiplication first. If we multiply the second equation by 3 and add the result to the first equation, we can eliminate y and solve for x, as shown at right.

$$\begin{array}{r} x+3y=4 \\ +\ \ 9x-3y=6 \\ \hline 10x \qquad =10 \\ x=1 \end{array}$$

We can then find y by substituting $x=1$ into either of the original equations. Choosing the second, we get $3(1)-y=2$, which solves to yield $y=1$. Again, checking the solution can be done by substituting both values into the first equation.

Answer: The solution to this system is $(1,1)$.

Now we can return to the original problem.

Solve the following system of linear equations in two variables.

$$5x - 4y = 7$$
$$2y + 6x = 22$$

For this system, you can use either the Substitution or the Elimination Method, but each choice will require a little bit of work to get started.

Substitution Method:

Before we can substitute, we need to isolate one of the variables. In other words, we need to solve one of the equations for either x or for y. If we solve the second equation for y, it becomes $y = 11 - 3x$. Now we substitute $11 - 3x$ for y in the first equation and solve for x, as shown at right.

$$\begin{aligned} 5x - 4(11 - 3x) &= 7 \\ 5x - 44 + 12x &= 7 \\ 17x - 44 &= 7 \\ 17x &= 51 \\ x &= 3 \end{aligned}$$

Then we can substitute the value for x into one of the original equations to find y. Thus we find that $2y + 6(3) = 22 \Rightarrow 2y = 22 - 18 = 4 \Rightarrow y = \frac{4}{2} = 2$.

Elimination Method:

Before we can eliminate a variable, we need to rearrange the second equation so that the variables line up, as shown at right. Now we see that we can multiply the second equation by 2 and add the two equations to eliminate y and solve for x, as shown below right.

$$\begin{aligned} 5x - 4y &= 7 \\ 6x + 2y &= 22 \end{aligned}$$

$$\begin{array}{rr} 5x - 4y &= 7 \\ +\ 12x + 4y &= 44 \\ \hline 17x \qquad &= 51 \\ x &= 3 \end{array}$$

We can then substitute $x = 3$ into the first equation to get $5(3) - 4y = 7$. Simplifying and solving, we get $-4y = -8$ and thus $y = 2$.

Answer: $(3, 2)$

Here are some more to try. Find the solution to these systems of linear equations. Use the method of your choice.

1. $y = 3x - 1$
   $2x - 3y = 10$

2. $x = -0.5y + 4$
   $8x + 3y = 31$

3. $2y = 4x + 10$
   $6x + 2y = 10$

4. $3x - 5y = -14$
   $x + 5y = 22$

5. $4x + 5y = 11$
   $2x + 6y = 16$

6. $x + 2y = 5$
   $x + y = 5$

7. $2x - 3 = y$
   $x - y = -4$

8. $y + 2 = x$
   $3x - 3y = x + 14$

9. $2x + y = 7$
   $x + 5y = 12$

10. $y = \frac{3}{5}x - 2$
    $y = \frac{x}{10} + 1$

11. $2x + y = -2x + 5$
    $3x + 2y = 2x + 3y$

12. $4x - 3y = -10$
    $x = \frac{1}{4}y - 1$

13. $4y = 2x$
    $2x + y = \frac{x}{2} + 1$

14. $3x - 2y = 8$
    $4y = 6x - 5$

15. $4y = 2x - 4$
    $3x + 5y = -3$

16. $\frac{x}{3} + \frac{4y}{3} = 300$
    $3x - 4y = 300$

Answers:

1. $(-1, -4)$
2. $(\frac{7}{2}, 1)$
3. $(0, 5)$
4. $(2, 4)$
5. $(-1, 3)$
6. $(5, 0)$
7. $(7, 11)$
8. $(-8, -10)$
9. $(\frac{23}{9}, \frac{17}{9})$
10. $(6, 1.6)$
11. $(1, 1)$
12. $(-\frac{1}{4}, 3)$
13. $(\frac{1}{2}, \frac{1}{4})$
14. no solution
15. $(\frac{4}{11}, -\frac{9}{11})$
16. $(300, 150)$

Checkpoint 3A

Problem 3-67

Expressions with Integral and Rational Exponents

Answers to problem 3-67: a: $x^{1/5}$, b: x^{-3}, c: $\sqrt[3]{x^2}$, d: $x^{-1/2}$, e: $\frac{1}{xy^8}$, f: $\frac{1}{m^3}$, g: $xy^3\sqrt{x}$,

h: $\frac{1}{81x^6y^{12}}$

The following properties are useful for rewriting expressions with integral (positive or negative whole numbers) or rational (fractional) exponents.

$x^0 = 1$ Examples: $2^0 = 1$, $(-3)^0 = 1$, $(\frac{1}{4})^0 = 1$ (Note that 0^0 is undefined.)

$x^{-n} = \frac{1}{x^n}$ Examples: $x^{-12} = \frac{1}{x^{12}}$, $y^{-4} = \frac{1}{y^4}$, $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

$\frac{1}{x^{-n}} = x^n$ Examples: $\frac{1}{x^{-5}} = x^5$, $\frac{1}{x^{-2}} = x^2$, $\frac{1}{3^{-2}} = 3^2 = 9$

$x^{a/b} = (x^a)^{1/b} = (\sqrt[b]{x})^a$ Examples: $5^{1/2} = \sqrt{5}$

or : $16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$

$x^{a/b} = (x^{1/b})^a = (\sqrt[b]{x})^a$ $4^{2/3} = \sqrt[3]{4^2} = \sqrt[3]{16} = 2\sqrt[3]{2}$

$x^a x^b = x^{(a+b)}$ Examples: $x^7x^2 = x^9$, $y^{-4}y = y^{-3}$, $2^32^2 = 2^5 = 32$

$(x^a)^b = x^{ab}$ Examples: $(x^2)^3 = x^6$, $(a^6b^4)^{1/2} = a^3b^2$, $(3^3)^3 = 3^9 = 19683$

Now we can go back and solve the original problems.

a. Using the fourth property above, $\sqrt[5]{x} = x^{1/5}$.

b. Using the second property above, $\frac{1}{x^3} = x^{-3}$.

c. Using the fourth property above, $x^{2/3} = \sqrt[3]{x^2}$.

d. Using the second and fourth properties above, $\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$.

e. Using the second property above, $x^{-1}y^{-8} = \frac{1}{xy^8}$.

f. Using the second and sixth properties above, $(m^2)^{-3/2} = m^{-3} = \frac{1}{m^3}$.

g. Using the fourth, fifth, and sixth properties above, $(x^3y^6)^{1/2} = x^{3/2}y^3 = x^1x^{1/2}y^3 = xy^3\sqrt{x}$.

h. Using the second and sixth properties above, $(9x^3y^6)^{-2} = 9^{-2}x^{-6}y^{-12} = \frac{1}{81x^6y^{12}}$.

Here are some exercises to try. For problems 1 through 12, rewrite each expression. For problems 13 through 24, simplify each expression. You should not need a calculator for any of these problems.

1. x^{-5}	2. m^0	3. 4^{-1}
4. $\sqrt[3]{y}$	5. $\frac{1}{c^4}$	6. $\frac{1}{b^{-2}}$
7. $12^{1/12}$	8. $z^{-3/4}$	9. $\frac{1}{(\sqrt[6]{7})^5}$
10. 0^0	11. $9^{1/2}$	12. $\sqrt[5]{a^3}$
13. $(f^3)\sqrt[3]{f^3}$	14. $(\frac{1}{27})^{-1/3}$	15. $(v^2 g^{3/4})^8$
16. $(\frac{1}{q^6})^7$	17. $d^{-9}d^{-4}$	18. $(3xw^4)^{-2}$
19. $(u^3 r^{-4})^{-2}$	20. $n^3(n^2)^5$	21. $4(\sqrt{4})^4$
22. $6(k^{1/2}t^5)^2$	23. $p^{15}p^{-15}$	24. $h^8 s^{12}(\sqrt[8]{h})(s^{1/4})$

Answers:

1. $\frac{1}{x^5}$	2. 1	3. $\frac{1}{4}$
4. $y^{1/3}$	5. c^{-4}	6. b^2
7. $\sqrt[12]{12}$	8. $\frac{1}{z^{3/4}}$	9. $7^{-1/6}$
10. undefined	11. 3	12. $a^{3/5}$
13. f^4	14. 3	15. $v^{16}g^6$
16. q^{-42}	17. $d^{-13} = \frac{1}{d^{13}}$	18. $\frac{1}{9x^2w^{12}}$
19. $\frac{r^8}{u^6}$	20. n^{13}	21. 64
22. $6kt^{10}$	23. 1	24. hs^3

Checkpoint 3B

Problem 3-116

Using Function Notation and Identifying Domain and Range

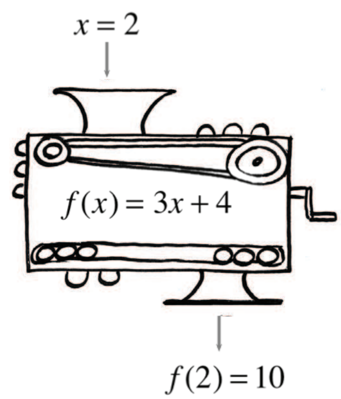
Answers to problem 3-116: Domain: all x; Range: $y \geq 0$

a. $g(-5) = 8$ b. $g(a+1) = 2a^2 + 16a + 32$

c. $x = 1$ or $x = -7$ d. $x = -3$

An equation is called a function if there exists *no more than one* output for each input. If an equation has two or more outputs for a single input value, it is not a function. The set of possible inputs of a function is called the domain, while the set of all possible outputs of a function is called the range.

Functions are often given names, most commonly "*f*," "*g*," or "*h*." The notation $f(x)$ represents the output of a function, named *f*, when *x* is the input. It is pronounced "*f* of *x*." The notation *g*(2), pronounced "*g* of 2," represents the output of function *g* when $x = 2$.

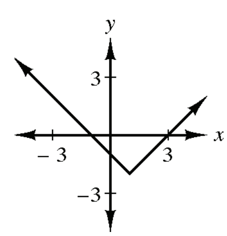


Similarly, the function $y = 3x + 4$ and $f(x) = 3x + 4$ represent the *same function*. Notice that the notation is interchangeable, that is $y = f(x)$. In some textbooks, $3x + 4$ is called the **rule** of the function. The graph of $f(x) = 3x + 4$ is a line extending forever in both the *x* (horizontal) and the *y* (vertical) directions, so the domain and range of $f(x) = 3x + 4$ are all real numbers.

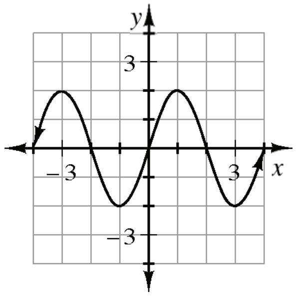
Examples 1 through 3: For each function below, give the domain and range. Then calculate $f(2)$ and solve $f(x) = 3$.

Example 1: $f(x) = |x - 1| - 2$

Solution: We start by graphing the function, as shown at right. Since we can use any real number for *x* in this equation, the domain is all real numbers. The smallest possible result for y is –2, so the range is $y \geq -2$. By looking at the graph or substituting $x = 2$ into the equation, $f(2) = |2 - 1| - 2 = -1$. To solve $f(x) = 3$, find the points where the horizontal line $y = 3$ intersects the graph or solve the equation $3 = |x - 1| - 2$, which yields $x = -4$ or $x = 6$.



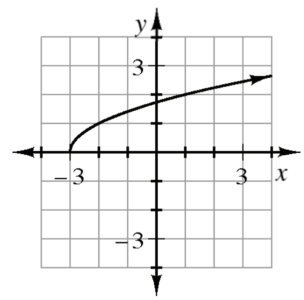
Example 2: $f(x)$ is given by the graph below.



The arrows indicate that the graph continues indefinitely right and left and we see no disruption in the smooth function, so the domain is all real numbers. All of the y-values fall between -2 and 2, so the range is $-2 \le y \le 2$. We can see from the graph that when $x = 2$, the value of the function appears to be 0, or $f(2) \approx 0$. Since $-2 \le y \le 2$, the value of the function never gets as high as 3, so $f(x) = 3$ has no solution.

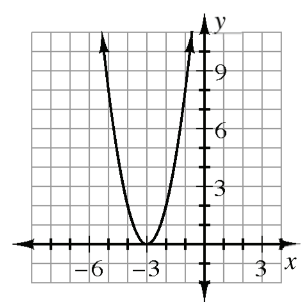
Example 3: $f(x) = \sqrt{x+3}$

Solution: Again, we start by making a graph of the function, which is shown at right. Since the square root of a negative number does not exist, we can only use x-values of -3 or larger. Thus, the domain is $x \ge -3$. We can see from the graph and the equation that the smallest possible y-value is zero, so the range is $y \ge 0$. Looking at the graph gives an approximate answer when $x = 2$ of $y \approx 2.25$. Or, by substituting $x = 2$ into the equation, we get $f(2) = \sqrt{2+3} = \sqrt{5}$. To solve $f(x) = 3$, find the point where $y = 3$ intersects the graph or solve $3 = \sqrt{x+3}$, which gives $x = 6$.



Now we can go back and solve the original problem.

The graph is a parabola opening upward with vertex $(-3,0)$, as shown at right. Thus, the domain is all real numbers and the range is $y \ge 0$.



$g(-5) = -2(-5+3) = 2(-2)^2 = 8$

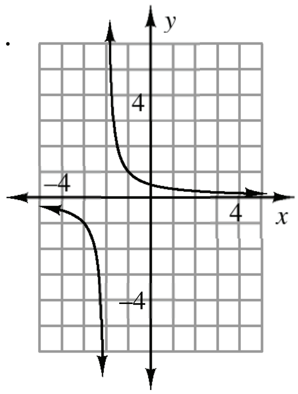
$$g(a+1) = 2(a+1+3)^2 = 2(a+4)^2$$
$$= 2(a^2 + 8a + 16) = 2a^2 + 16a + 32$$

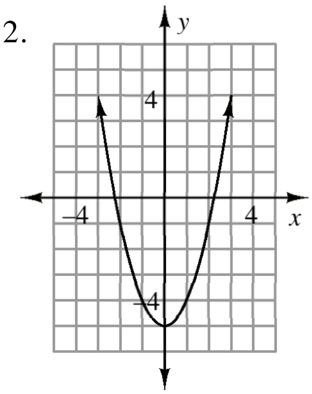
If $g(x) = 32$, then $32 = 2(x+3)^2$. Dividing both sides by 2, we get $16 = (x+3)^2$. Taking the square root of both sides gives $\pm 4 = x + 3$, which leads to the values $x = 1$ or -7.

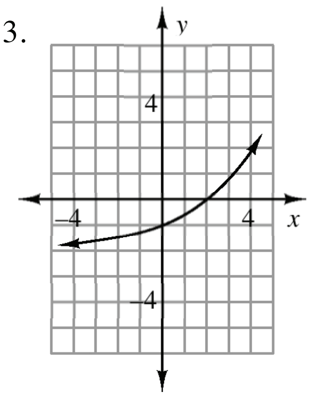
If $g(x) = 0$, then $0 = 2(x+3)^2$. Diving both sides by two or applying the Zero Product Property gives $0 = (x+3)^2$ and then $0 = x + 3$. Thus $x = -3$.

Here are some more to try.

For each graph in problems 1 through 3, describe the domain and range.

1. 

2. 

3. 

4. If $f(x) = 3 - x^2$, calculate $f(5)$ and $f(3a)$.

5. If $g(x) = 5 - 3x^2$, calculate $g(-2)$ and $g(a+2)$.

6. If $f(x) = \frac{x+3}{2x-5}$, calculate $f(2)$ and $f(2.5)$.

7. If $f(x) = x^2 + 5x + 6$, solve $f(x) = 0$.

8. If $g(x) = 3(x-5)^2$, solve $g(x) = 27$.

9. If $f(x) = (x+2)^2$, solve $f(x) = 27$.

Answers:

1. Domain: $x \neq -2$, Range: $y \neq 0$

2. Domain: all real numbers, Range: $y \geq -5$

3. Domain: all real numbers, Range: $y > -2$

4. $f(5) = -22$, $f(3a) = 3 - 9a^2$

5. $g(-2) = -7$, $g(a+2) = -3a^2 - 12a - 7$

6. $f(2) = -5$, not possible

7. $x = -2$ or $x = -3$

8. $x = 8$ or $x = 2$

9. $x = -2 \pm \sqrt{27}$

Checkpoint 4A

Problem 4-42

Writing Equations for Arithmetic and Geometric Sequences

Answers to problem 4-42: a: E $t(n)=-2+3n$; R $t(0)=-2,\ t(n+1)=t(n)+3$,
b: E $t(n)=6(\frac{1}{2})^n$; R $t(0)=6,\ t(n+1)=\frac{1}{2}t(n)$, c: $t(n)=10-7n$, d: $t(n)=5(1.2)^n$,
e: $t(4)=1620$

An ordered list of numbers such as: $4, 9, 16, 25, 36, \ldots$ creates a sequence. The numbers in the sequence are called terms. One way to identify and label terms is to use function notation. For example, if $t(n)$ is the name of the sequence above, the initial value is 4 and the second term after the initial value is 16. This is written $t(0)=4$ and $t(2)=16$. Some books use subscripts instead of function notation. In this case $t_0=4$ and $t_2=16$. The initial value is *not* part of the sequence. It is only a reference point and is useful in writing a rule for the sequence. When writing a sequence, start by writing the first term after the initial value, $t(1)$ or t_1 .

Arithmetic sequences have a common difference between the terms. The rule for the values in an arithmetic sequences can be found by $t(n)=a+dn$ where a = the initial value, d = the common difference and n = the number of terms after the initial value.

Geometric sequences have a common ratio between the terms. The rule for the values in a geometric sequence may be found by $t(n)=ar^n$ where a = the initial value, r = the common ratio and n = the number of terms after the initial value.

Example 1: Find a rule for the sequence: –2, 4, 10, 16, …

Solution: There is a common difference between the terms $(d=6)$ so it is an arithmetic sequence. Work backward to find the initial value: $a=-2-6=-8$.
Now use the general rule: $t(n)=a+dn=-8+6n$.

Example 2: Find a rule for the sequence: 81, 27, 9, 3, …

Solution: There is a common ratio between the terms $(r=\frac{1}{3})$ so it is a geometric sequence.
Work backward to find the initial value: $a=81\div\frac{1}{3}=243$.
Now use the general rule: $t(n)=ar^n=243(\frac{1}{3})^n$.

A rule such as $t(n)=5-7n$ is called an explicit rule because any term can be found by substituting the appropriate value for n into the rule. To find the 10$^{\text{th}}$ term after the initial value, $t(10)$, substitute 10 for n. $t(10)=5-7(10)=-65$.

A second way to find the terms in a sequence is by using a recursive formula. A recursive formula tells first term or the initial value and how to get from one term to the next.

Example 3: (using subscript notation)
Write the first five terms of the sequence determined by: $b_1 = 8$, $b_{n+1} = b_n \cdot \frac{1}{2}$

Solution: $b_1 = 8$ tells you the first term and $b_{n+1} = b_n \cdot \frac{1}{2}$ tells you to multiply by $\frac{1}{2}$ to get from one term to the next.

$b_1 = 8$ $\quad$ $b_2 = b_1 \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4$ $\quad$ $b_3 = b_2 \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2$

$b_4 = b_3 \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$ $\quad$ $b_5 = b_4 \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$

The sequence is: $8, 4, 2, 1, \frac{1}{2}, \ldots$

Now we can go back and solve the original problems.

a. It is an arithmetic sequence $(d = 3)$. Working backward the initial value is $1 - 3 = -2$. Using the general formula the explicit rule: $t(n) = a + dn = -2 + 3d$. A possible recursive rule is $t(1) = 1, t(n+1) = t(n) + 3$.

b. It is a geometric sequence $(r = \frac{1}{2})$. Working backward the initial value is $3 \div \frac{1}{2} = 6$. Using the general formula for the explicit rule: $t(n) = ar^n = 6(\frac{1}{2})^n$. A possible recursive rule is $t(0) = 6, t(n+1) = \frac{1}{2}t(n)$.

c. $t(2)$ is halfway between $t(1)$ and $t(3)$ so $t(2) = 10$. This means $d = -7$ and the initial value is 24. Using the general formula the explicit rule: $t(n) = a + dn = 24 - 7d$.

d. The common ratio $r = \frac{8.64}{7.2} = 1.2$ so $t(1) = \frac{7.2}{1.2} = 6, t(0) = \frac{6}{1.2} = 5$. Using an initial value of 5 and a common ratio of 1.2 in the general formula for the explicit rule: $t(n) = ar^n = 5(1.2)^n$.

e. The common difference is the difference in the values divided by the number of terms. $d = \frac{t(12) - t(7)}{12 - 7} = \frac{116 - 1056}{5} = -188$. Working backward three terms: $t(4) = 1056 - 3(-188) = 1620$.

Here are some more to try.

Write the first 6 terms of each sequence.

1. $t(n) = 5n + 2$
2. $t(n) = 6(-\frac{1}{2})^n$
3. $t(n) = -15 + \frac{1}{2}n$
4. $t_n = -3 \cdot 3^{n-1}$
5. $t(1) = 3,\ t(n+1) = t(n) - 5$
6. $t_1 = \frac{1}{3},\ t_{n+1} = \frac{1}{3}t_n$

For each sequence, write an explicit and recursive rule.

7. 10, 50, 250, 1250, …
8. 4, 8, 12, 16, …
9. –2, 5, 12, 19, …
10. 16, 4, 1, $\frac{1}{4}$, ...
11. –12, 6, –3, $\frac{3}{2}$, …
12. $\frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$, ...

For each sequence, write an explicit rule.

13. A geometric sequence

n	$t(n)$
0	
1	15
2	45
3	
4	

14. An arithmetic sequence

n	$t(n)$
0	27
1	15
2	
3	
4	

15. An arithmetic sequence

n	$t(n)$
1	
2	$3\frac{1}{3}$
3	
4	
5	$4\frac{1}{3}$

16. A geometric sequence

n	$t(n)$
1	
2	
3	–24
4	48
5	

Solve each problem.

17. An arithmetic sequence has $t(3)=52$ and $t(10)=108$. Find a rule for $t(n)$ and find $t(100)$.

18. An arithmetic sequence has $t(1)=-17$, $t(2)=-14$ and $t(n)=145$. What is the value of n?

19. An arithmetic sequence has $t(61)=810$ and $t(94)=1239$. Find a rule for $t(n)$.

20. A geometric sequence has $t(4)=12$ and $t(7)=324$. Find the common ratio and a rule for $t(n)$.

Answers:

1. $7, 12, 17, 22, 27, 32$
2. $-3, \frac{3}{2}, -\frac{3}{4}, \frac{3}{8}, -\frac{3}{16}, \frac{3}{32}$
3. $-14\frac{1}{2}, -14, -13\frac{1}{2}, -13, -12\frac{1}{2}, -12$
4. $-3, -9, -27, -81, -243, -729$
5. $3, -2, -7, -12, -17, -22$
6. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}$

Rules for problems 7 through 20 may vary.

7. $t(n)=2\cdot 5^n;\quad t(0)=2,\ t(n+1)=5t(n)$
8. $t(n)=4n;\quad t(0)=0,\ t(n+1)=t(n)+4$
9. $t(n)=-9+7n;\quad t(0)=-9,\ t(n+1)=t(n)+7$
10. $t(n)=64(\frac{1}{4})^n;\quad t(0)=64,\ t(n+1)=\frac{1}{4}t(n)$
11. $t(n)=24(-\frac{1}{2})^n;\quad t(0)=24,\ t(n+1)=-\frac{1}{2}t(n)$
12. $t(n)=1-\frac{1}{6}n;\quad t(0)=1,\ t(n+1)=t(n)-\frac{1}{6}$
13. $t(n)=5\cdot 3^n$
14. $t(n)=27-12n$
15. $t(n)=2\frac{2}{3}+\frac{1}{3}n$
16. $t(n)=3(-2)^n$
17. $t(n)=28+8n;\ t(100)=828$
18. $n=55$
19. $t(n)=17+13n$
20. $t(n)=\frac{4}{27}(3)^n$

Checkpoint 4B

Problem 4-87

Solving for One Variable in an Equation with Two or More Variables

Answers to problem 4-87: a: $y = \frac{1}{3}x - 4$, b: $y = \frac{6}{5}x - \frac{1}{5}$, c: $y = (x+1)^2 + 4$, d: $y = x^2 + 4x$

When we want to solve for one variable in an equation with two or more variables it usually helps to start by simplifying, such as removing parentheses and fractions. Next isolate the desired variable in the same way as you solve an equation with only one variable. Here are two examples.

Example 1: Solve $\frac{x-3y}{4} + 2(x+1) = 7$ for *y*.

Solution: First multiply all terms by 4 to remove the fraction and then simplify, as shown at right. Then, to isolate *y*, we subtract 9*x* from both sides to get $-3y = -9x + 20$. Dividing both sides by –3 results in $y = 3x - \frac{20}{3}$.

$$(4)\frac{x-3y}{4} + 4(2)(x+1) = 4(7)$$
$$x - 3y + 8x + 8 = 28$$
$$9x - 3y = 20$$

Answer: $y = 3x - \frac{20}{3}$

Example 2: Solve $x + 2\sqrt{y+1} = 3x + 4$ for *y*.

Solution: First, we isolate the radical by subtracting *x* from both sides to get $2\sqrt{1+y} = 2x + 4$ and then dividing both sides by 2 to get $\sqrt{1+y} = x + 2$. Then, we remove the radical by squaring both sides, as shown at right. Lastly, we isolate *y* by subtracting 1 from both sides of the equation.

$$(\sqrt{y+1})^2 = (x+2)^2$$
$$y + 1 = (x+2)(x+2)$$
$$y + 1 = x^2 + 4x + 4$$

Answer: $y = x^2 + 4x + 3$

Now we can go back and solve the original problems.

a.
$$\begin{aligned} x-3(y+2)&=6 \\ x-3y-6&=6 \\ x-3y&=12 \\ -3y&=-x+12 \\ y&=\tfrac{-x+12}{-3} \text{ or } y=\tfrac{1}{3}x-4 \end{aligned}$$

b.
$$\begin{aligned} \tfrac{6x-1}{y}-3&=2 \\ \tfrac{6x-1}{y}&=5 \\ (y)\tfrac{6x-1}{y}&=5(y) \\ 6x-1&=5y \\ y&=\tfrac{6x-1}{5} \text{ or } y=\tfrac{6}{5}x-\tfrac{1}{5} \end{aligned}$$

c.
$$\begin{aligned} \sqrt{y-4}&=x+1 \\ (\sqrt{y-4})^2&=(x+1)^2 \\ y-4&=(x+1)^2 \\ y&=(x+1)^2+4 \text{ or } x^2+2x+5 \end{aligned}$$

d.
$$\begin{aligned} \sqrt{y+4}&=x+2 \\ (\sqrt{y+4})^2&=(x+2)^2 \\ y+4&=x^2+4x+4 \\ y&=x^2+4x \end{aligned}$$

Here are some more to try. Solve each equation for *y*.

1. $2x-5y=7$
2. $2(x+y)+1=x-4$
3. $4(x-y)+12=2x-4$
4. $x=\frac{1}{5}y-2$
5. $x=y^2+1$
6. $\frac{5x+2}{y}-1=5$
7. $\sqrt{y+3}=x-2$
8. $(y+2)^2=x^2+9$
9. $\frac{x+2}{4}+\frac{4-y}{2}=3$
10. $\sqrt{2y+1}=x+3$
11. $x=\frac{2}{4-y}$
12. $x=\frac{y+1}{y-1}$

Answers:

1. $y=\frac{2}{5}x-\frac{7}{5}$
2. $y=-\frac{1}{2}x-\frac{5}{2}$
3. $y=\frac{1}{2}x+4$
4. $y=5x+10$
5. $y=\pm\sqrt{x-1}$
6. $y=\frac{5}{6}x+\frac{1}{3}$
7. $y=x^2-4x+1$
8. $y=\pm\sqrt{x^2+9}-2$
9. $y=\frac{1}{2}x-1$
10. $y=\frac{1}{2}x^2+3x+4$ or $y=\frac{1}{2}(x+4)(x+2)$
11. $y=\frac{4x-2}{x}$ or $y=4-\frac{2}{x}$
12. $y=\frac{x+1}{x-1}$

Checkpoint 5A

Problem 5-49

Multiplying Polynomials

Answers to problem 5-49:

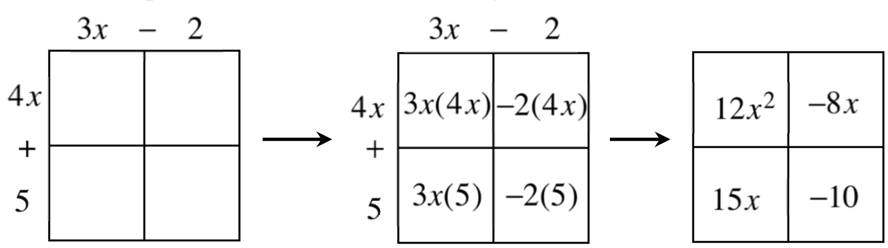
a. $2x^3 + 2x^2 - 3x - 3$ b. $x^3 - x^2 + x + 3$

c. $2x^2 + 12x + 18$ d. $4x^3 - 8x^2 - 3x + 9$

The product of polynomials can be found by using the Distributive Property. Using generic rectangles or, in the case of multiplying two binomials, the FOIL Method can help you to keep track of the terms to be sure that you are multiplying correctly.

Example: Multiply $(3x-2)(4x+5)$.

Solution 1: When multiplying binomials, such as $(3x-2)(4x+5)$, you can use a generic rectangle. You consider the terms of your original binomials as the dimensions (length and width) of the rectangle. To find the area of each piece, you multiply the terms that represent the length and width of that piece. To get your final answer, you add the areas of each of the interior pieces and simplify by combining like terms. This process is shown in the diagram below.



$12x^2 + 15x - 8x - 10 = 12x^2 + 7x - 10$

Solution 2: You might view multiplying binomials with generic rectangles as a form of double distribution. The $4x$ is distributed across the first row of the generic rectangle. Then the 5 is distributed across the second row of the generic rectangle. Some people write it this way:

$$(3x-2)(4x+5) = (3x-2)4x + (3x-2)5 = 12x^2 - 8x + 15x - 10 = 12x^2 + 7x - 10$$

Solution 3: Another approach to multiplying binomials is to the FOIL Method. This method uses the mnemonic "FOIL," which is an acronym for First, Outside, Inside, Last, to help you remember which terms to multiply.

F.	Multiply the FIRST terms of each binomial.	$(3x)(4x) = 12x^2$
O.	Multiply the OUTSIDE terms.	$(3x)(5) = 15x$
I.	Multiply the INSIDE terms.	$(-2)(4x) = -8x$
L.	Multiply the LAST terms of each binomial.	$(-2)(5) = -10$

Finally combine like terms to get $12x^2 + 15x - 8x - 10 = 12x^2 + 7x - 10$.

Answer: $12x^2 + 7x - 10$

Now we can go back and solve the original problems.

a: $(x+1)(2x^2 - 3)$

Solution: We can use the FOIL Method here. Multiplying the *first* terms, we get $(x)(2x^2) = 2x^3$. Multiplying the *outside* terms, we get $(x)(-3) = -3x$. Multiplying the *inside* terms, we get $(1)(2x^2) = 2x^2$. Multiplying the *last* terms, we get $(1)(-3) = -3$. Adding these results, we get $2x^3 - 3x + 2x^2 - 3$. Generally, answers are expressed in with terms in order of decreasing powers of *x*, so we rearrange terms for the answer.

Answer: $2x^3 + 2x^2 - 3x - 3$

b: $(x+1)(x - 2x^2 + 3)$

Solution: This is a good problem for a generic rectangle, as shown at right. After calculating the area of each individual cell, we find our expression by adding them together to get $x^3 - 2x^2 + x^2 + 3x - 2x + 3$. Then we combine like terms to get a simplified answer.

	x^2	$-2x$	$+3$
x	x^3	$-2x^2$	$3x$
$+1$	x^2	$-2x$	3

Answer: $x^3 - x^2 + x + 3$

c: $2(x+3)^2$

Solution: Here we write out the factors and use the Distributive Property, as shown in the solution at right.

$$\begin{aligned} &2(x+3)(x+3) \\ &= (2x+6)(x+3) \\ &= (2x+6)(x)+(2x+6)(3) \\ &= 2x^2+6x+6x+18 \\ &= 2x^2+12x+18 \end{aligned}$$

Answer: $2x^2+12x+18$

d: $(x+1)(2x-3)^2$

Solution: Write out the factors. Multiply two of the factors together and then multiply that result by the third factor. This process is shown at right.

$$\begin{aligned} &(x+1)(2x-3)(2x-3) \\ &= (2x^2-x-3)(2x-3) \\ &= 4x^3-6x^2-2x^2+3x-6x+9 \\ &= 4x^3-8x^2-3x+9 \end{aligned}$$

Answer: $4x^3-8x^2-3x+9$

Here are some more to try. Multiply and simplify.

1. $(2x+3)(x-7)$
2. $(4x-2)(3x+5)$
3. $(x-2)(x^2+3x+5)$
4. $(x+8)(x-12)$
5. $4(3x-5)^2$
6. $(2x+y)(2x-y)$
7. $(2x+3)^2$
8. $(5x-8)(2x+7)$
9. $(x+3)(x^2-4x+7)$
10. $(x+7)(x-11)$
11. $-8x^2(5x^2+7)$
12. $(2x+y)(x+1)^2$

Answers:

1. $2x^2-11x-21$
2. $12x^2+14x-10$
3. x^3+x^2-x-10
4. $x^2-4x-96$
5. $36x^2-120x+100$
6. $4x^2-y^2$
7. $4x^2+12x+9$
8. $10x^2+19x-56$
9. $x^3-x^2-5x+21$
10. $x^2-4x-77$
11. $-40x^4-56x^2$
12. $2x^3+4x^2+2x+x^2y+2xy+y$

Checkpoint 5B

Problem 5-100

Factoring Quadratics

Answers to problem 5-100:

a. $(2x+1)(2x-1)$ b. $(2x+1)^2$

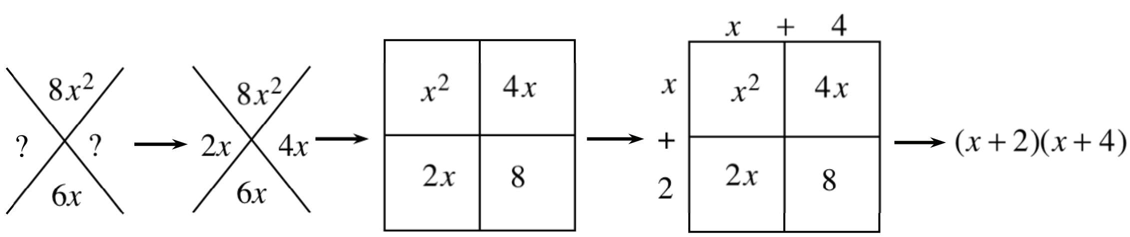
c. $(2y+1)(y+2)$ d. $(3m+1)(m-2)$

Factoring quadratics means changing the expression into a product of factors or to find the dimensions of the generic rectangle that represents the quadratic. You can use Diamond Problems with generic rectangles or just guess and check with FOIL Method or the Distributive Property to factor.

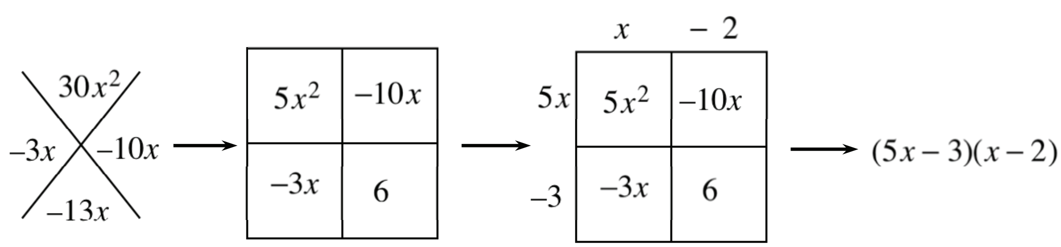
Here are some examples using Diamond Problems and generic rectangles:

Example 1: Factor x^2+6x+8.

Solution: Multiply the x^2-term by the constant term and place the result in the top of the diamond. This will be the product of the two sides of the diamond. Then place the x-term at the bottom of the diamond. This will be the sum of the sides. Then find two terms that multiply to give the top term in the diamond and add to give the bottom term in the diamond, in this case $2x$ and $6x$. This tells us how the x-term is split in the generic rectangle. Once we have the area of the generic rectangle we can find the dimensions by looking for common factors among rows and columns. Study the example below.

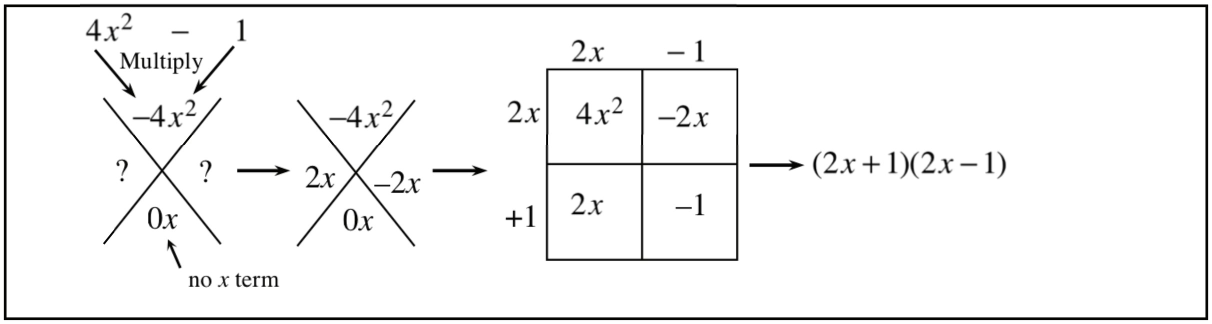


Example 2: Factor $5x^2-13x+6$.

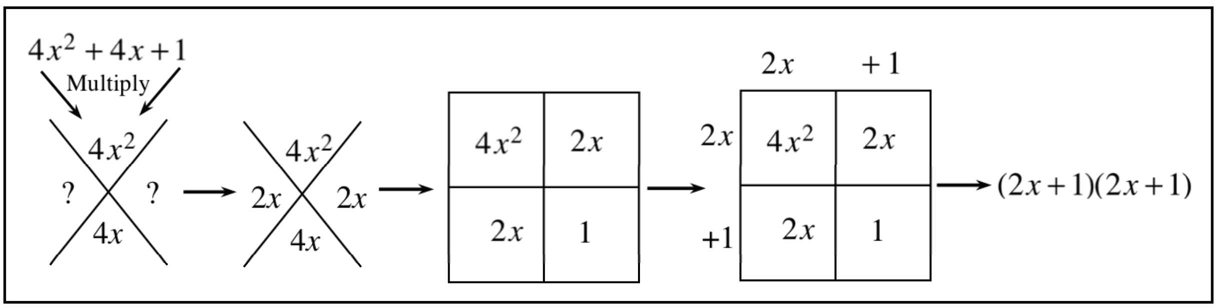


Now we can go back and solve the original problems.

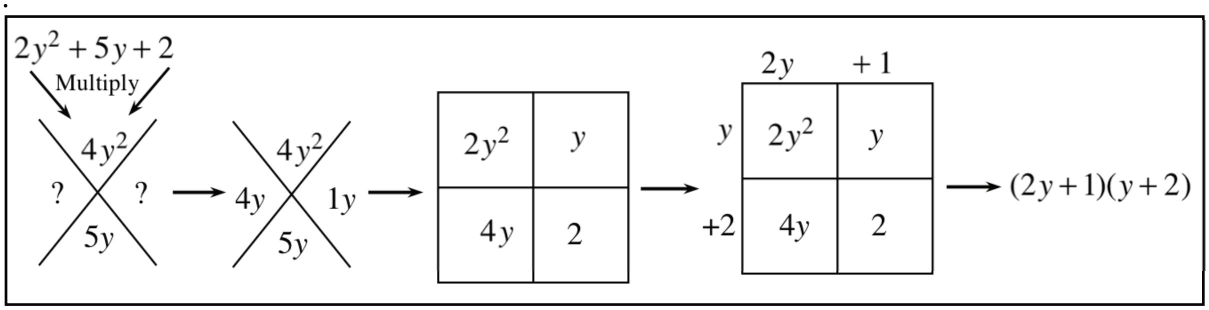
a.



b.



c.



d.